Quantum mechanics on rational numbers

A. Vourdas
University of Bradford

- background
  - $\mathbb{Q}_p/\mathbb{Z}_p$ and its Pontryagin dual group $\mathbb{Z}_p$
  - $\mathbb{Q}/\mathbb{Z}$ and its Pontryagin dual group $\widehat{\mathbb{Z}}$
  - $\mathbb{Q}$ and its Pontryagin dual group $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$
    $\mathbb{Q}^{(\pi)}$ and its Pontryagin dual $\mathbb{S}_\pi$ (solenoid)
    $n^{-1}\mathbb{Z}$ and its Pontryagin dual $\mathbb{R}/n\mathbb{Z}$
- The Schwartz-Bruhat space for $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$
- examples: $S(\mathbb{S}_\pi, \mathbb{Q}^{(\pi)})$, $S[(\mathbb{R}/n\mathbb{Z}), n^{-1}\mathbb{Z}]$
- Heisenberg-Weyl group and other phase space methods in $(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Q}$
- the set of subsystems of $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$
  - partial order
  - T_0-topology
- Discussion

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$\mathbb{Q}_p/\mathbb{Z}_p$ and its Pontryagin dual group $\mathbb{Z}_p$

- $\mathbb{Z}_p$ as inverse limit.

$$\varprojlim \mathbb{Z}(p^\ell) = \mathbb{Z}_p$$

  $\mathbb{Z}_p$ profinite: compact, totally disconnected group

$$a_p = \overline{a}_0 + \overline{a}_1 p + ...; \quad 0 \leq \overline{a}_i \leq p - 1$$

- Pontryagin dual group of $\mathbb{Z}_p$ is $\mathbb{Q}_p/\mathbb{Z}_p$
  $\mathbb{Q}_p/\mathbb{Z}_p$ as direct limit.

$$\varinjlim \mathbb{Z}(p^\ell) = \mathbb{Q}_p/\mathbb{Z}_p$$

  fractional p-adic numbers (cosets)

$$\mathfrak{b}_p = \overline{\mathfrak{b}}_{-k} p^{-k} + ... + \overline{\mathfrak{b}}_{-1} p^{-1}$$

  $\mathbb{Q}_p/\mathbb{Z}_p$ isomorphic to Prüfer group $\mathcal{C}(p^\infty)$

- characters in $\mathbb{Q}_p/\mathbb{Z}_p$

$$\chi_p(a_p \mathfrak{b}_p) = \exp(i 2\pi a_p \mathfrak{b}_p)$$

$\mathbb{Q}/\mathbb{Z}$ and its Pontryagin dual group $\widehat{\mathbb{Z}}$

- $\widehat{\mathbb{Z}}$ as inverse limit

$$\varprojlim \mathbb{Z}(\ell) = \widehat{\mathbb{Z}}$$

  $\widehat{\mathbb{Z}}$ profinite group

$$\widehat{\mathbb{Z}} = \prod_{p \in \Pi} \mathbb{Z}_p$$

  elements

$$s = (s_2, ..., s_p, ...); \quad s_p \in \mathbb{Z}_p; \quad p \in \Pi$$

  $\mathbb{Z}$ is embedded into $\widehat{\mathbb{Z}}$:

$$\mathbb{Z} \ni n \quad \rightarrow \quad (n, n, n, ...) \in \widehat{\mathbb{Z}}$$

- Pontryagin dual group of $\widehat{\mathbb{Z}}$ is $\mathbb{Q}/\mathbb{Z}$
  $\mathbb{Q}/\mathbb{Z}$ as direct limit:

$$\varinjlim \mathbb{Z}(\ell) = \mathbb{Q}/\mathbb{Z} \qquad \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \Pi} \mathbb{Q}_p/\mathbb{Z}_p$$

  $\mathfrak{x} \in \mathbb{Q}/\mathbb{Z}$: $(\mathfrak{x}_2, ..., \mathfrak{x}_p, ...)$, where $\mathfrak{x}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, $p \in \Pi$
  all but a finite number of the $\mathfrak{x}_p$ zero

- characters in $\mathbb{Q}/\mathbb{Z}$

$$\chi(s\mathfrak{x}) = \prod_{p \in \Pi} \chi_p(s_p \mathfrak{x}_p)$$

  converges: only a finite number of the $\chi_p(s_p \mathfrak{x}_p)$ different from 1.

$\mathbb{Q}$ and its Pontryagin dual group $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$

- ring $\mathbb{A}_\mathbb{Q}$ of adeles

  $y = (y_\infty, y_2, ..., y_p, ...); \qquad y_p \in \mathbb{Q}_p; \qquad \mathbb{Q}_\infty = \mathbb{R}$

  but $y_p \in \mathbb{Z}_p$ for all but a finite number of p
  $\mathbb{A}_\mathbb{Q}$: **restricted** direct product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$:

$$\mathbb{A}_\mathbb{Q} = \prod{}' \mathbb{Q}_p$$

- $\mathbb{Q}$ embedded into $\mathbb{A}_\mathbb{Q}$ as

$$\mathbb{Q} \ni u \quad \to \quad (u, u, u, ...) \in \mathbb{A}_\mathbb{Q}$$

- $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ as inverse limit

$$\varprojlim \mathbb{R}/n\mathbb{Z} \cong \mathbb{A}_\mathbb{Q}/\mathbb{Q}$$

  $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ describes finite covers of $\mathbb{S}$.

- Pontryagin dual group of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is $\mathbb{Q}$
  $\mathbb{Q}$ as direct limit

$$\varinjlim n^{-1}\mathbb{Z} \cong \mathbb{Q}$$

- Additive characters on $\mathbb{Q}$ are given by

$$\psi(uy) = \text{exp}[i2\pi(-uy_\infty + uy_2 + ...)] = \prod_{p \in \Pi_\infty} \chi_p(uy_p)$$

$$u \in \mathbb{Q}; \qquad y \in \mathbb{A}_\mathbb{Q}/\mathbb{Q}$$
$$\chi_\infty(uy_\infty) = \text{exp}(-i2\pi uy_\infty)$$

  minus sign in $\chi_\infty(uy_\infty)$ and a plus sign in $\chi_p(uy_p)$
  $y \;\to\; y+$rational: same result

  convergence: finite number of factors $\neq 1$

- fundamental domain for $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is $\mathbb{S} \times \widehat{\mathbb{Z}}$
  ($\mathbb{S} = \mathbb{R}/\mathbb{Z}$).

- $\mathbb{Q}^{(\pi)}$: subgroup of $\mathbb{Q}$

$$\mathbb{Q}^{(\pi)} = \left\{ \frac{a}{\prod_{p_i \in \pi} p_i^{e_i}} \mid a \in \mathbb{Z} \right\}$$

Pontryagin dual group of $\mathbb{Q}^{(\pi)}$: solenoid $\mathbb{S}_\pi$
$(y_0, y_2, ..., y_p, ...)$ where $y_p = 0$ for $p \notin \pi$

$$\varprojlim \mathbb{R}/k^n\mathbb{Z} \cong \mathbb{S}_\pi; \quad k = \prod_{p \in \pi} p^e$$

special case: $\pi = \{p\}$:
p-adic solenoid $\mathbb{S}_p$ with elements (y_0, y_p).

- subgroup of $\mathbb{Q}^{(\pi)}$

$$n^{-1}\mathbb{Z} = \left\{ \frac{a}{n} = \frac{a}{\prod_{p_i \in \pi} p_i^{e_i}} \mid e_i \leq E_i \right\}; \quad n = \prod_{p \in \pi} p_i^{E_i}$$

Pontryagin dual group $\mathbb{R}/n\mathbb{Z}$

elements of $\mathbb{R}/n\mathbb{Z}$: (y_0, k) where $y_0 \in \mathbb{S}$ and $k \in \mathbb{Z}(n)$ winding number
$\mathbb{Z}(n) \cong \prod_{p \in \pi} \mathbb{Z}(p_i^{E_i})$
chinese remainder theorem: winding number $k = (y_{p_1}, ...)$ where $y_{p_i} \in \mathbb{Z}(p_i^{E_i})$

elements of $\mathbb{R}/n\mathbb{Z}$: $(y_0, y_2, ..., y_p, ...)$
$y_{p_i} \in \mathbb{Z}(p_i^{E_i})$ component of winding number for that prime

The Schwartz-Bruhat space for $S[(\mathbb{A}_\mathbb{Q}/\mathbb{Q}),\mathbb{Q}]$

first: space Σ functions on $\mathbb{A}_\mathbb{Q}$
later: space $\mathfrak{S}$ functions on $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$

- Schwartz-Bruhat space Σ: finite linear combinations of complex functions $\phi(y)$

  $$\phi(y) = \phi_\infty(y_\infty)\prod_{p\in\Pi}\phi_p(y_p); \qquad y = (y_\infty, y_2, ..., y_p, ...) \in \mathbb{A}_\mathbb{Q}$$

  where

  (1) $\phi_\infty(y_\infty) \in \mathcal{S}(\mathbb{R})$ (Schwartz)

  (2) $\phi_p(y_p)$ locally constant complex functions with compact support,

  (3) for all but a finite number of $p \in \Pi$
  $\phi_p(y_p) = 1$ if y_p is p-adic integer

  $\Pi[\phi(y)]$ contains indices $\phi_p(y_p) \neq 1$
  $\Pi_1[\phi(y)]$ subset $y_p \in \mathbb{Q}_p$ (finite)
  $\Pi_2[\phi(y)]$ subset $y_p \in \mathbb{Z}_p$ (finite)

- Integrals of functions in Σ over $\mathbb{A}_\mathbb{Q}$:

  $$\int_{\mathbb{A}_\mathbb{Q}}\phi(y)dy = \int_\mathbb{R}\phi_\infty(y_\infty)dy_\infty$$
  $$\times \prod_{p\in\Pi_1[\phi(y)]}\int_{\mathbb{Q}_p}\phi_p(y_p)dy_p \prod_{p\in\Pi_2[\phi(y)]}\int_{\mathbb{Z}_p}\phi_p(y_p)dy_p$$

  finite number of factors $\neq 1$
  p-adic integrals of locally constant functions with constant support = finite sums.

- space $\mathfrak{S}$
  From $\phi(y)$ on $\mathbb{A}_{\mathbb{Q}}$, to $f(\mathfrak{y})$ on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$
  Weil transf: add values of $\phi(y)$ in each coset in $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$:

  $$f(\mathfrak{y}) = \int_{\mathbb{Q}} du\, \phi(y+u); \qquad \mathfrak{y} = \{y+u \mid u \in \mathbb{Q}\}; \qquad \mathfrak{y} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$$

  then

  $$\int_{\mathbb{A}_{\mathbb{Q}}} \phi(y) dy = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} f(\mathfrak{y}) d\mathfrak{y}$$

- Fourier transform of $f(\mathfrak{y})$

  $$F(u) = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} f(\mathfrak{y})\psi(u\mathfrak{y}) d\mathfrak{y} = \int_{\mathbb{A}_{\mathbb{Q}}} \phi(y)\psi(yu) dy; \qquad u \in \mathbb{Q}$$

  write $F(u)$ as

  $$F(u) = F_\infty(u) \prod_{p \in \Pi[\phi(y)]} F_p(u) \prod_{p \notin \Pi[\phi(y)]} \Delta_p(u)$$

  Fourier trans. of $\phi_p(y_p) = 1$, $y_p \in \mathbb{Z}_p$:

  $\Delta_p(u) = 0$ if $u \neq 0$
  $\Delta_p(0) = 1$ zero coset in $\mathbb{Q}_p/\mathbb{Z}_p$: p-adic integers

  $u = a/b$ is p-adic integer if $p \not| b$
  $\Delta_p(u) = 1$ if $u = a/b$ with $p \not| b$

- inverse Fourier transform

$$f(\mathfrak{y}) = \int_{\mathbb{Q}} du\ F(u)\ \psi(-u\mathfrak{y}); \qquad \mathfrak{y} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$$

- scalar product

$$(f,g) = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} \overline{f(\mathfrak{y})} g(\mathfrak{y}) d\mathfrak{y}; \quad (F,G) = \int_{\mathbb{Q}} du\ \overline{F(u)}\ G(u)$$

- Parseval: $(f,g) = (F,G)$

below examples with variables in subgroups of $\mathbb{Q}$
and in their Pontryagin dual groups

work in fundamental domain $\mathbb{S} \times \widehat{\mathbb{Z}}$ **of** $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$.

Examples

- QM for $S[(\mathbb{R}/\mathbb{Z}),\mathbb{Z}]$
  $f(\mathfrak{y}) = f_\infty(\mathfrak{y}_\infty)$ where $\mathfrak{y}_\infty \in \mathbb{S}$
  (i.e., $f_p(\mathfrak{y}_p) = 1$ for all $p \in \Pi$)
  Fourier transform

$$\begin{aligned} F(u) &= \int_{\mathbb{S}\times\widehat{\mathbb{Z}}} f_\infty(\mathfrak{y}_\infty)\psi(u\mathfrak{y})d\mathfrak{y} \\ &= \left[\int_0^1 d\mathfrak{y}_\infty f_\infty(\mathfrak{y}_\infty)\chi_\infty(u\mathfrak{y}_\infty)\right] \\ &\times \prod_{p\in\Pi} \Delta_p(u) \end{aligned}$$

  $F(u)$ non-zero: if $u \in \mathbb{Z}$
  i.e., $u = a/b$ with $p \not| b$ for **all** primes

- QM for $S(\mathbb{S}_{p_1},\mathbb{Q}^{(p_1)})$
  $f(\mathfrak{y}) = f_\infty(\mathfrak{y}_\infty)f_{p_1}(\mathfrak{y}_{p_1})$
  (i.e., $f_p(\mathfrak{y}_p) = 1$ for all $p \in \Pi - \{p_1\}$)
  Fourier transform

$$\begin{aligned} F(u) &= \int_{\mathbb{S}\times\widehat{\mathbb{Z}}} f_\infty(\mathfrak{y}_\infty)f_{p_1}(\mathfrak{y}_{p_1})\psi(u\mathfrak{y})d\mathfrak{y} \\ &= \left[\int_0^1 d\mathfrak{y}_\infty f_\infty(\mathfrak{y}_\infty)\chi_\infty(u\mathfrak{y}_\infty)\int_{\mathbb{Z}_p} d\mathfrak{y}_{p_1} f_{p_1}(\mathfrak{y}_{p_1})\chi_{p_1}(u\mathfrak{y}_{p_1})\right] \\ &\times \prod_{p\in\Pi-\{p_1\}} \Delta_p(u) \end{aligned}$$

  $F(u)$ non-zero only if $u \in \mathbb{Q}^{(p_1)}$
  i.e., $u = a/b$ with $p \not| b$ for $p \in \Pi - \{p_1\}$

- QM for $S[(\mathbb{R}/p_1^{e_1}\mathbb{Z}), p_1^{-e_1}\mathbb{Z}]$
  restrict the above formalism further
  $f_{p_1}(\mathfrak{y}_{p_1})$ locally constant with **given degree** e_1:
  $f_{p_1}(\mathfrak{y}_{p_1} + \mathfrak{a}_{p_1}) = f_{p_1}(\mathfrak{y}_{p_1})$ for $|\mathfrak{a}_{p_1}|_{p_1} \leq p_1^{-e_1}$

  $\mathfrak{y}$: pair $(\mathfrak{y}_\infty, \mathfrak{y}_{p_1})$ with $\mathfrak{y}_\infty \in \mathbb{R}/\mathbb{Z}$ and $\mathfrak{y}_{p_1} \in \mathbb{Z}(p_1^{e_1})$
  describes points in circle $\mathbb{R}/(p_1^{e_1}\mathbb{Z})$ wrapped $p_1^{e_1}$
  times around the circle $\mathbb{R}/\mathbb{Z}$

- QM for $S(\mathbb{S}_\pi, \mathbb{Q}^{(\pi)})$ with $\pi = \{p_1, ..., p_\ell\}$

  $$f(\mathfrak{y}) = f_\infty(\mathfrak{y}_\infty) f_{p_1}(\mathfrak{y}_{p_1}) ... f_{p_\ell}(\mathfrak{y}_{p_\ell})$$

  Fourier transform $F(u)$ non-zero only if
  $u \in \mathbb{Q}^{(\pi)}$

- QM for $S[(\mathbb{R}/n\mathbb{Z}), n^{-1}\mathbb{Z}]$ with $n = p_1^{e_1} ... p_\ell^{e_\ell}$
  restrict the formalism further
  $f_{p_j}(\mathfrak{y}_{p_j})$, locally constant, with given degrees e_j

Phase space methods in $(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Q}$

- The Heisenberg-Weyl group $\mathcal{HW}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}, \mathbb{Q}, \mathbb{A}_{\mathbb{Q}}/\mathbb{Q})$ displacement operators $D(\mathfrak{a}, b, \mathfrak{c})$

$$
\begin{aligned}
&[\mathcal{D}(\mathfrak{a}, b, \mathfrak{c})F](u) = \psi\,(\mathfrak{c} - \mathfrak{a}b + 2\mathfrak{a}u)\, F(u-b) \\
&[\mathcal{D}(\mathfrak{a}, b, \mathfrak{c})f](\mathfrak{y}) = \psi\,(\mathfrak{c} + \mathfrak{a}b - \mathfrak{y}b)\, f(\mathfrak{y} - 2\mathfrak{a}) \\
&\mathfrak{a}, \mathfrak{c}, \mathfrak{y} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}; \quad\ b, u \in \mathbb{Q}
\end{aligned}
$$

  $\mathcal{HW}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}, \mathbb{Q}, \mathbb{A}_{\mathbb{Q}}/\mathbb{Q})$ locally compact topological group.

- For any trace class operator θ

$$
\int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a} \int_{\mathbb{Q}} db\, \mathcal{D}(\mathfrak{a}, b, 0)\, \theta\, [\mathcal{D}(\mathfrak{a}, b, 0)]^{\dagger} = \mathbf{1}\mathrm{tr}\theta
$$

$$
\theta = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a} \int_{\mathbb{Q}} db\, \mathcal{D}(\mathfrak{a}, b, 0)\mathrm{tr}[\theta\mathcal{D}(-\mathfrak{a}, -b, 0)]
$$

- coherent states

$$
f_{\mathrm{coh}}(\mathfrak{y}|\mathfrak{a}, b) \equiv [\mathcal{D}(\mathfrak{a}, b, \mathfrak{c})f](\mathfrak{y}); \qquad \mathfrak{a} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}; \qquad b \in \mathbb{Q}
$$

  with resolution of the identity:

$$
\int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a} \int_{\mathbb{Q}} db\, f_{\mathrm{coh}}(\mathfrak{y}|\mathfrak{a}, b)\, \overline{f_{\mathrm{coh}}(\mathfrak{y}'|\mathfrak{a}, b)} = \delta_A(\mathfrak{y} - \mathfrak{y}')
$$

- parity around origin: $\mathcal{P}(0,0)F(u) = F(-u)$
parity around $(\mathfrak{a}, b) \in (\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Q}$

$$\mathcal{P}(\mathfrak{a}, b) = \mathcal{D}(\mathfrak{a}, b, \mathfrak{c})\ \mathcal{P}(0,0)\ [\mathcal{D}(\mathfrak{a}, b, \mathfrak{c})]^{\dagger}$$

Parity related to displacements with Fourier tr

$$\mathcal{P}(\mathfrak{a}, b) = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a}' \int_{\mathbb{Q}} db'\ \mathcal{D}(\mathfrak{a}', b', 0)\psi(2\mathfrak{a}'b - 2\mathfrak{a}b')$$

operator θ can be expanded as

$$\theta = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a} \int_{\mathbb{Q}} db\ \mathcal{P}(\mathfrak{a}, b,)\mathrm{tr}[\theta \mathcal{P}(\mathfrak{a}, b)]$$

- Given a pair of functions $g(\mathfrak{y}), f(\mathfrak{y}) \in \mathfrak{S}$
Wigner $\mathcal{W}(\mathfrak{a}, b; g, f) = (g, \mathcal{P}(\mathfrak{a}, b)f)$
Weyl (or ambiguity) $\widetilde{\mathcal{W}}(\mathfrak{a}, b; g, f) = (g, \mathcal{D}(\mathfrak{a}, b, 0)f)$

$$\mathcal{W}(\mathfrak{a}, b; g, f) = \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a}' \int_{\mathbb{Q}} db'\ \widetilde{\mathcal{W}}(\mathfrak{a}', b'; g, f)\psi(2\mathfrak{a}'b - 2\mathfrak{a}b')$$

'marginal properties' of Wigner function

$$\begin{aligned} \int_{\mathbb{Q}} db\ \mathcal{W}(\mathfrak{a}, b; g, f) &= \overline{g(-2\mathfrak{a})}\ f(-2\mathfrak{a}) \\ \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a}\ \mathcal{W}(\mathfrak{a}, b; g, f) &= \overline{\widetilde{g}(-b)}\ \widetilde{f}(-b) \end{aligned}$$

'marginal properties' of Weyl function:

$$\begin{aligned} \int_{\mathbb{Q}} db\ \widetilde{\mathcal{W}}(\mathfrak{a}, b; g, f) &= \overline{g(\mathfrak{a})}\ f(-\mathfrak{a}) \\ \int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} d\mathfrak{a}\ \widetilde{\mathcal{W}}(\mathfrak{a}, b; g, f) &= \overline{\widetilde{g}(2^{-1}b)}\ \widetilde{f}(-2^{-1}b) \end{aligned}$$

the set of subsystems of $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}),\mathbb{Q}]$

- supernatural (Steinitz) numbers:

  $$\mathbb{N}_S = \left\{\prod p^{e_p} \mid p \in \Pi; \quad e_p \in \mathbb{Z}_0^+ \cup \{\infty\}\right\}$$

  examples:

  $$\Omega = \prod_{p\in\Pi} p^{\infty}; \quad \Omega(\pi) = \prod_{p\in\pi} p^{\infty}; \quad \Omega(\pi)|\Omega$$

  directed partially ordered set
  with divisibility as order: $m \prec n$ means $m|n$
  Ω 'top element'

- directed-complete partial order (dcpo): each chain has a supremum
  $\mathbb{N}_S$: dcpo
  $p, p^2, p^3, ..., p^\infty$ has p^∞ as supremum
  $m, m^2, ..., \Omega[\pi(m)]$ has $\Omega[\pi(m)]$ as supremum

- Topological space $(\mathbb{N}_S, T_{\mathbb{N}_s})$ with the 'divisor topology' $T_{\mathbb{N}_s}$ generated by the base

  $$B_{\mathbb{N}_s} = \{\emptyset, U(n) \mid n \in \mathbb{N}_S\}; \quad U(n) = \{m \in \mathbb{N}_S \mid m|n\}$$

  T_0 topological space (but not T_1)
  separation axioms: T_2 (Hausdorff)$\subset T_1 \subset T_0$

- partial order: $\mathbb{N}_S$: dcpo, $\mathbb{N}$ not dcpo
  topology: $\mathbb{N}_S$: compact, $\mathbb{N}$ locally compact, not compact

  $\mathbb{N}$: something is missing!!
  $\mathbb{N}_S$: we found what was missing!!

- subsystem: $S(E,\widetilde{E}) \prec S(G,\widetilde{G})$ if $\widetilde{E}$ subgroup of $\widetilde{G}$
  then quotient relation between E, G with annihilators

- A_S: set of subsystems of $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$
  bijective map between A_S and $\mathbb{N}_S$:
  $S[(\mathbb{R}/n\mathbb{Z}), n^{-1}\mathbb{Z}] \;\; \leftrightarrow \;\; n \in \mathbb{N}$
  $S(\mathbb{S}_\pi, \mathbb{Q}^{(\pi)}) \;\; \leftrightarrow \;\; \Omega(\pi)$
  $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}] \;\; \leftrightarrow \;\; \Omega$

  also $A = \{S[(\mathbb{R}/n\mathbb{Z}), n^{-1}\mathbb{Z}] \mid n \in \mathbb{N}\}$
  bijective map between A and $\mathbb{N}$

- A_S order isomorphic to $\mathbb{N}_S$

  embeddings of subsystems into supersystems and their **compatibility:**
  quantum states, Heisenberg-Weyl groups, Wigner functions, etc

  $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$ maximum element
  $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$: QM on ‘large circles’

  A not dcpo (something is missing!!)
  A_S dcpo (we found what was missing!!)

- (A_S, T_{A_S}) topological space homeomorphic to $(\mathbb{N}_S, T_{\mathbb{N}_S})$

  T_0-topology:
  axioms of T_0 topology express basic logical relations between subsystems and supersystems

  can be used to define continuity of a quantity (eg entropy) in systems and their supersystems

- A locally compact, not compact (something is missing!!)
  A_S compact (we found what was missing!!)

- $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$ smallest system that contains all $S(\mathbb{S}_\pi, \mathbb{Q}^{(\pi)})$, $S[(\mathbb{R}/q\mathbb{Z}), q^{-1}\mathbb{Z}]$ as subsystems

details for $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}]$: work in progress

details for $S[\widehat{\mathbb{Z}}, (\mathbb{Q}/\mathbb{Z})]$:
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Discussion

- The Schwartz-Bruhat space for $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}),\mathbb{Q}]$
- examples: $S(\mathbb{S}_\pi,\mathbb{Q}^{(\pi)})$, $S[(\mathbb{R}/n\mathbb{Z}),n^{-1}\mathbb{Z}]$
- Heisenberg-Weyl group and other phase space methods in $(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})\times\mathbb{Q}$
- partial order and topology of the set of subsystems of $S[(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}),\mathbb{Q}]$
  directed complete partial order
  topology: compact
  QM on large circles $\mathbb{R}/n\mathbb{Z}$