Induced representations of infinite-dimensional groups

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$$L^2(X, V, \mu) = \{f : X \to V \mid \|f\|^2 := \int_X \|f(x)\|_V^2 d\mu(x) < \infty\},$$

where $\mu = \mu_s$ is a G-quasi-invariant measure on X satisfying the condition $d\mu_s(xg)/d\mu_s(x) = \Delta_H(h(x,g))/\Delta_G(h(x,g))$. Here Δ_G is a *modular function* on a group G.

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The *induced representation* is defined by the following formula

$$(T(g)f)(x)=S(h(x,g))\,(d\mu(xg)/d\mu(x))^{1/2}f(xg), \qquad (1)$$

where $h(x,g)\in H$ is defined by formula $s(x)g=h(x,g)s(xg)$.

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Remark

The right (or the left) regular representation $\rho, \lambda : G \mapsto U(L^2(G, h))$ *of a locally compact group* G *is a particular case of the induced representation* $\mathrm{Ind}_H^G S$ *with* $H = \{e\}$ *and* $S = Id$*, where* h *is a Haar measure. The* quasiregular representation *is a particular case of the induced representation with some closed subgroup* $H \subset G$ *and* $S = Id$*.*

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b) the set $\mathcal{O}(G)$ of all orbits of the group G in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ with respect to the coadjoint representation.
A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *subordinate* to a functional $f \in \mathfrak{g}^*$ if

$$\langle f, [x,y]\rangle = 0 \quad for\ \ all \quad x, y \in \mathfrak{h},$$

i.e. if $\mathfrak{h}$ is an *isotropic subspace* with respect to the bilinear form defined by $B_f(x,y) = \langle f, [x,y]\rangle$ on $\mathfrak{g}$, where $\langle f, x\rangle = \mathrm{tr}(xf)$, $x \in \mathfrak{g}$, $f \in \mathfrak{g}^*$.

Orbit method for $B(n,\mathbb{R})$

One-dimensional representation of the Lie algebra $\mathfrak{h}$ is defined by $\mathfrak{h} \ni x \mapsto \langle f, x\rangle \in \mathbb{R}$. We define a one-dimensional unitary representation $U_{f,H} : H \to S^1$ of the group $H = \exp \mathfrak{h}$ by formula

$$U_{f,H}(\exp x) = \exp 2\pi i \langle f, x\rangle. \tag{2}$$

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Theorem (Theorem 7.2, [2])

(a) Every irreducible unitary representation T of G has the form $T=\mathrm{Ind}_H^G U_{f,H}$, where $H\subset G$ is a connected subgroup and $f\in\mathfrak{g}^*$;

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(c) two irreducible representations are equivalent $T_{f_1,H_1} \sim T_{f_2,H_2}$ if and only if the functionals f_1 and f_2 belong to the same orbit of $\mathfrak{g}^*$.

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Let us fix a Lie group $G = B(n,\mathbb{R})$, let $\mathfrak{g}$ be the its Lie algebra, and $\mathfrak{g}^*$ the dual space. The pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is defined by the trace:

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The *adjoint action* of the group G on $\mathfrak{g}$ has the following form $\mathrm{Ad}_t(x) = txt^{-1}$, $t \in G$, $x \in \mathfrak{g}$. The *coadjoint action* of the group G on $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{Ad}^*_t(y), x\rangle = \langle y, \mathrm{Ad}_t(x)\rangle,\ y \in \mathfrak{g}^*, x \in \mathfrak{g}$$

and is expressed as $\mathrm{Ad}^*_t(y) = (t^{-1}yt)_-$, where $(A)_-$ means that we take lower triangular part of A.

Generic orbits

The form of the action $\mathrm{Ad}^*_t(y) = (t^{-1}yt)_-$ implies, that Ad^*_t, $t \in G$ acts as follows: to a given column of $y \in \mathfrak{g}^*$, a linear combination of the previous columns is added and to a given row of y, a linear combination of the following rows is added.

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The minors Δ_k, $k = 1, 2, ..., [\frac{n}{2}]$, consisting of the last k rows and first k columns of y are invariant of the action. It is possible to show that If $c_k \neq 0$ then the manifold given by the equation

$$\mathcal{O}_{c_1,c_2,\dots,c_{[n/2]}} = \left\{y \in \mathfrak{g}^* \mid \Delta_k = c_k,\ 1 \leq k \leq [n/2]\right\}$$

is a G-orbit in $\mathfrak{g}^*$.

Generic orbits

Hence, *generic orbits* $\mathcal{O}_{c_1,c_2,\dots,c_{[n/2]}}$ have codimension equal to $[\frac{n}{2}]$ and dimension equal to $\frac{n(n-1)}{2}-[\frac{n}{2}]$.

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$$y^{n+1}=\left(\begin{smallmatrix}0&0\\ \Lambda&0\end{smallmatrix}\right)=\sum_{r+s=n+1,\,1\leq s\leq[n/2]} y_{rs}E_{rs},\quad y^5=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&y_{32}&0&0\\y_{41}&0&0&0\end{pmatrix},$$

where Λ is the matrix of order $[\frac{n}{2}]$ such that all nonzero elements are contained in the *anti-diagonal*.

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where Λ is the matrix of order $[\frac{n}{2}]$ such that all nonzero elements are contained in the *anti-diagonal*. A subalgebra subordinate to the functional y consists of all matrices of the form $\left(\begin{smallmatrix}0&A\\0&0\end{smallmatrix}\right)$, where A is an $[\frac{n}{2}]\times[\frac{n+1}{2}]$ or $[\frac{n+1}{2}]\times[\frac{n}{2}]$ matrix.

Regular and quasiregular representations, $\dim G = \infty$

(a) "Regular representation". Find a suitable topological group $\tilde{G}$:
1) $G \subset \tilde{G}$ and G is a dense subgroup in $\tilde{G}$,
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Representation $T^{R,\mu} : G \mapsto U(L^2(\tilde{G}, \mu))$ is defined by
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The right regular representation $T^{R,\mu} : G \to U(L^2(\tilde{G},\mu))$ is irreducible if and only if
1) $\mu^{L_t} \perp \mu \ \forall t \in G\backslash\{e\}$, ($\perp$ means singular),
2) the measure μ is G-ergodic.

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3) in $L^2(\tilde{X},V,\mu)=\{f:\tilde{X}\to V : \int_{\tilde{X}}\|f(x)\|_V^2 d\mu(x)<\infty,\}$ define the induced representation of the group G by the following formula:

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where $\tilde{h}$ is defined by $\tilde{s}(x)t=\tilde{h}(x,t)\tilde{s}(xt)$ for an appropriate section $\tilde{s}:\tilde{X}\to\tilde{G}$ of the extended projection $\tilde{p}:\tilde{G}\to\tilde{X}$.

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$\sigma_2(a)=\{x=\sum_{k,n\in\mathbb{Z}}x_{kn}E_{kn}\mid \|x\|^2_{\sigma_2(a)}=\sum_{k,n\in\mathbb{Z}}\mid x_{kn}\mid^2 a_{kn}<\infty\}$,

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$\mathfrak{A}_{\mathrm{GL}}=\{a=(a_{kn})_{(k,n)\in\mathbb{Z}^2} \mid 0<a_{kn}\leq Ca_{km}a_{mn},\ k,n,m\in\mathbb{Z}, C>0\}$.
Define *Hilbert-Lie algebra* $\mathfrak{g}l_2(a)$ and *Hilbert-Lie group* $\mathrm{GL}_2(a)$ as:
$\mathfrak{g}l_2(a)=\{x=\sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn} \mid \|x\|^2_{\mathfrak{g}l_2(a)}=\sum_{k,n\in\mathbb{Z}} \mid x_{kn} \mid^2 a_{kn}< \infty\}$,
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Theorem ([3])

Every continuous unitary representation U of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ in a Hilbert space H can be extended by continuity to a unitary representation $U_2(a):\mathrm{GL}_2(a)\to U(H)$ of some Hilbert-Lie group $\mathrm{GL}_2(a)$, $a\in\mathfrak{A}_{\mathrm{GL}}$ depending on the representation.

Consider a Hilbert-Lie group $B_2(a) := \{I + x \mid x \in \mathfrak{b}_2(a)\}$, the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a)$ is defined as

$$\mathfrak{b}_2(a) = \{x = \sum_{(k,n)\in\mathbb{Z}^2, k<n} x_{kn} E_{kn} \mid \|x\|^2_{\mathfrak{b}_2(a)} = \sum_{(k,n)\in\mathbb{Z}^2, k<n} \mid x_{kn} \mid^2 a_{kn} < \infty\},$$

$$\mathfrak{A} = \left\{a = (a_{kn})_{(k,n)\in\mathbb{Z}^2, k<n},\ a_{kn} \leq C a_{km} a_{mn},\ k < m < n,\ k, m, n \in \mathbb{Z}\right\}.$$

Consider a Hilbert-Lie group $B_2(a) := \{I + x \mid x \in \mathfrak{b}_2(a)\}$, the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a)$ is defined as

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Hence, for the description of *the dual space* $\widehat{B_0^{\mathbb{Z}}}$ it is sufficient to know $\widehat{B_2(a)}$ for all $a \in \mathfrak{A}$, but this *problem has not been solved yet*.

The space of orbits

Take the group $B_0^{\mathbb{Z}}$, fix one of its Hilbert–Lie completion $B_2(a)$, $a \in \mathfrak{A}$, the corresponding Hilbert–Lie algebra $\mathfrak{b}_2(a)$ and the dual space $\mathfrak{b}_2^*(a)$ w.r.t the pairing

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Case 1) 0-dimensional orbits are of the form: $\mathcal{O}_0 = y,\ y \in \mathfrak{b}_2^*(a)$, $y = \sum_{k\in\mathbb{Z}} y_{k+1,k}E_{k+1,k}$. $B_2(a) \ni \exp(x) \mapsto \exp(2\pi i(\langle y, x\rangle)) \in S^1$.

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Case 2) The finite-dimensional orbits corresponding to *finite points* $y = \sum_{(k,n)\in\mathbb{Z},k>n} y_{kn}E_{kn} \in \mathfrak{b}_2^*(a)$. Kirillov: $\hat{G} \supset \cup_n \hat{G}_{2n-1}$.

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Case 4) General orbits generated by points $y \in \mathfrak{b}_2^*(a)$. Rep-s.??

Recall $B_0^{\mathbb{Z}} \subset B_2(a) \subset B^{\mathbb{Z}}$ and $\mathfrak{b}_0^{\mathbb{Z}} \subset \mathfrak{b}_2(a) \subset \mathfrak{b}^{\mathbb{Z}}$. Fix $y^k \in (\mathfrak{b}_0^{\mathbb{Z}})^*$, the Lie algebra $\mathfrak{h}_0^{2m+1} = \{\sum_{r\leq m<n} x_{rn}E_{rn}\} \subset \mathfrak{b}_0^{\mathbb{Z}}$ is subordinate to the functional y^k for all $k, m \in \mathbb{Z}$ since it is commutative. The representation of the Lie algebra

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We have $G = B_0^{\mathbb{Z}}$ and $S : H \mapsto U(\mathbb{C})$. Construct $\mathrm{Ind}_H^G(S)$?
1) Extension of the representations $\tilde{S} : \tilde{H} \mapsto U(V)$.
2) Completion of the space $\tilde{X} = \tilde{H}\backslash\tilde{G}$.
3) Construction of the G-quasiinvariant measure on $\tilde{X}$.

1) The representation S of the group H_0^{2m+1} can be extended to the representation of its Hilbert–Lie completion $H_2^{2m+1}(a) \subset B_2(a)$ for some $a \in \mathfrak{A}$: $H_2^{2m+1}(a) \ni \exp(x) \longmapsto \exp 2\pi i \langle y^k, x\rangle \in S^1$. Indeed, $\mathfrak{b}_0^{\mathbb{Z}} = \cap_{a\in\mathfrak{A}} \mathfrak{b}_2(a)$, therefore $(\mathfrak{b}_0^{\mathbb{Z}})^* = \cup_{a\in\mathfrak{A}} \mathfrak{b}_2^*(a)$, hence any $y^k \in (\mathfrak{b}_0^{\mathbb{Z}})^*$ belongs to some $\mathfrak{b}_2^*(a)$, $a \in \mathfrak{A}$.

2) Completion of the space $X = H\backslash G$. For $m \in \mathbb{Z}$ we have
$B_0^{\mathbb{Z}} = B_{m,0} B_0(m) B_0^{(m)}$, $B_2(a) = B_{m,2}(a) B_2(m, a) B_2^{(m)}(a)$,
$B^{\mathbb{Z}} = B_m B(m) B^{(m)}$, hence $X_{m,0} \subset X_{m,2}(a) \subset X_m$, where
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Lemma (Kolmogorov's zero-one law.)

We have $\mu_b(B_{m,2}(a) \times B_2^{(m)}(a)) = 1$ *(resp.* $= 0$*) if and only if*

$$\sum_{(k,n)\in B_m \cup B^{(m)}} a_{kn}/b_{kn} < \infty \quad (\textit{resp.} \quad = \infty).$$

Irreducibility criteria

The representation $T^{y^k,2m+1,\mu_b}$, $k,m\in\mathbb{Z}$, is defined by

$$(T_t f)(x)=S(h(x,t))\,(d\mu(xt)/d\mu(x))^{1/2}\,f(xt),\;\; f\in L^2(X,\mu),$$

$$S(h(x,t))=\exp 2\pi i\langle y,h(x,t)-I\rangle=\exp\left(2\pi i\,\mathrm{tr}\left((t-I)B(x,y)\right)\right). \quad (6)$$

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(ii) The representation $T^{2m,2m+2r+1,\mu_b}$ is irreducible if and only if (a) the measure μ_b is $B_0^{\mathbb{Z}}$ ergodic and (b) either $r=-1$, $r=0$ or $r<-1$ and $\mu_b^{L_t}\perp\mu_b$ for all $t\in G_{m-|r|+1,m+|r|-1}\backslash\{e\}$.

We find $h(x,t)$ using $s(x)t{=}h(x,t)s(xt)$. Set $B(x{,}y){=}x_m^{-1}yx^{(m)}$, we get

$$h(x,t)-I=\begin{cases} 0, & \text{for} \quad t\in B_mB^{(m)}, \\ x^{(m)}(t-I)x_m^{-1}, & \text{for} \quad t\in B(m), \end{cases}$$

$$\langle y, h(x,t){-}I\rangle = \mathrm{tr}\left(x^{(m)}t_0x_m^{-1}y\right) = \mathrm{tr}\left(t_0x_m^{-1}yx^{(m)}\right) = \mathrm{tr}\left(t_0B(x,y)\right).$$

The group $B^{\mathbb{Z}}$ is a *semi-direct product* $B^{\mathbb{Z}} = B_m \ltimes B(m) \rtimes B^{(m)}$, we have $B_mB(m)B^{(m)} \ni x_mx(m)x^{(m)}{=}hx_mx^{(m)} \in B(m)B_mB^{(m)}$, $h{=}x_mx(m)x_m^{-1}$, where $B^{\mathbb{Z}} \ni x = \begin{pmatrix} x^{(m)} & x(m) \\ 0 & x_m \end{pmatrix} = x_mx(m)x^{(m)}$. The space $X = B(m)\backslash B^{\mathbb{Z}}$ is isomorphic to $B_mB^{(m)}$. Therefore the section s can be used as an embedding $s\colon B_mB^{(m)}{\to}B(m)B_mB^{(m)}$. For $t = t_mt^{(m)} \in B_mB^{(m)}$ holds $h(x,t) = e$. For $t \in B(m)$ we get $s(x)t{=}x_mx^{(m)}t{=}h(x,t)x_mx^{(m)}$, hence $h(x{,}t){=}x_mx^{(m)}t(x_mx^{(m)})^{-1}$ $= \begin{pmatrix} x^{(m)} & 0 \\ 0 & x_m \end{pmatrix}\begin{pmatrix} 1 & t_0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (x^{(m)})^{-1} & 0 \\ 0 & x_m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x^{(m)}t_0x_m^{-1} \\ 0 & 1 \end{pmatrix}$, where $t_0{=}t{-}I$.

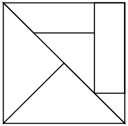
Proof of the theorem

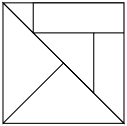
For the particular case $k=2m+1$, $r=0$ or $k=2m$, $r=0$ or $r=-1$ the statement of the theorem is exactly the same as in the finite-dimensional case. The proof of the irreducibility is based on

Lemma

The von Neumann algebra $\mathfrak{A}^S$ generated by the restriction of the representation $T^{y^k,2m+2r+1,\mu_b}$ on the commutative subgroup $B_0(m)$ of the group $B_0^{\mathbb{Z}}$ coincides with $L^\infty(X_m,\mu_b)$.

$$G_{p,q}=\{I+\textstyle\sum_{p\le k<r\le q} x_{kr}E_{kr}\},\ B(m)=H^{2m+1}=\{I+\textstyle\sum_{k\le m<r} x_{kr}E_{kr}\}.$$





$$B^{\mathbb{Z}}\ni x=\begin{pmatrix} x^{(m)} & x(m)\\ 0 & x_m\end{pmatrix}.$$

Define the unitary representation $T^{L,2n+1,\mu_b}$, $n \in \mathbb{Z}$ of the group $G = B_{n,0} \times B_0^{(n)}$ in the Hilbert space $\mathcal{H} = L^2(B_n \times B^{(n)}, \mu_b)$ by

$$(T_s^{L,2n+1,\mu_b} f)(x) = \left(d\mu_b(s^{-1}x)/d\mu_b(x)\right)^{1/2} f(s^{-1}x),\ f \in \mathcal{H},\ s \in G,$$

where $\mu_b = \mu_{b,n} \otimes \mu_b^{(n)}$ is defined by (4).

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where $\mu_b = \mu_{b,n} \otimes \mu_b^{(n)}$ is defined by (4). The representation $T_s^{L,2n+1,\mu_b}$ is correctly defined for any $s \in B_0^{(n)}$ and for an arbitrary measure $\mu_{b,n}$.

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$$\mu_{b,n}^{L_I+tE_{rs}}\sim\mu_{b,n},\ \forall t\in\mathbb{R}\Leftrightarrow S_{rs}^L(b)=\sum_{n=s+1}^{\infty}b_{rn}/b_{sn}<\infty.$$

Resume, conclusions

What we can say about $\hat{G}$ for $G = B_0^{\mathbb{Z}}$?

We have $B_0^{\mathbb{Z}} = \cap_{a\in\mathfrak{A}} B_2(a)$, therefore $\widehat{B_0^{\mathbb{Z}}} = \cup_{a\in\mathfrak{A}} \widehat{B_2(a)}$.

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