Harmonic Analysis for Locally Compact Abelian Groups

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Unterstürtzt von/Supported by the Alexander von Humboldt Stiftung/Foundation

p-Adic Methods for Modelling of Complex Systems
18 April 2013

Outline

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Meta-Harmonic Analysis, I

- $f : G \to \mathbb{C}$, G a locally compact abelian group (LCAG)
- Consider $f \mapsto F(y) = \int_G f(x)\psi_y(x)d\mu(x)$.
- Especially desirable when $f = \int \left[\int_G f(x)\psi_y(x)d\mu(x)\right] \tilde{\psi}_y d\tilde{\mu}(y)$ (weakly, pointwise, etc.).

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- Time-Frequency Analysis: $f \in L^2(G)$, $\psi_{\gamma,y} = \chi_\gamma T_y \phi$, $\chi_\gamma \in \hat{G}$, $y \in G$, $T_y h(\cdot) = h(\cdot - y)$, $\|\phi\|_{L^2} = 1$, $F(\gamma, y) \in L^2(G \times \hat{G})$. (quantum mechanics, continuous Short-Time Frequency Transform, Gabor-Weyl-Heisenberg)

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- Time-Scale Analysis: $f \in L^2(G)$, $\psi_{a,b} = D_a T_b \phi$, ϕ "nice" in $L^2(G)$, $a \neq 0$, $b \in G$, $D_a h(\cdot) = |a|^{1/2} h(a\cdot)$. (continuous wavelet transform)

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Some key questions about a discretely indexed system $\Psi = \{\psi_i\}_i$ in a function space $\mathbf{F}$:

- Is Ψ an *orthonormal basis*?
- Is Ψ a *frame*? That is, do there exist $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$ (generalizations possible to Banach spaces),

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_i |\langle f, \psi_i \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2.$$

- Is Ψ *complete* in $\mathbf{F}$ (a normed vector space)? That is, is it dense?

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  - Shifting by a set of coset representatives of $\mathbb{Q}_p^k/\mathbb{Z}_p^k$ (Maria's talk).
  - Something else which has a group structure. (Next slide)

"Benedetto & Benedetto" Shifts – Notation

- G a LCAG, $H < G$ a compact open subgroup.
- The *annihilator subgroup* of H in $\hat{G}$ is defined to be

$$H^{\perp} = \{\gamma \in \hat{G} : \forall x \in H, \quad \chi_{\gamma}(x) = 1\} \subseteq \hat{G},$$

- $\widehat{G/H} \cong H^{\perp}$ and $\hat{H} \cong \hat{G}/H^{\perp}$
- Haar measures normalized so measure of H in G is one, the measure of $H^{\perp}$ in $\hat{G}$ is one, and the measures on G/H and $\hat{G}/H^{\perp}$ are the counting measure.

"Benedetto & Benedetto" Shifts – Definition

Definition (Benedetto & Benedetto 2004)

Let G be a LCAG with compact open subgroup $H \leq G$. Let $\mathcal{D} \subseteq \hat{G}$ be a set of coset representatives in $\hat{G}$ for the quotient $\hat{H} = \hat{G}/H^{\perp}$.

- Define maps $\theta = \theta_{\mathcal{D}} : \hat{G} \to \mathcal{D}$ and $\eta = \eta_{\mathcal{D}} : \hat{G} \to H^{\perp} \subseteq \hat{G}$ by
  - $\theta(\gamma) =$ the unique $\sigma_\gamma \in \mathcal{D}$ such that $\gamma - \sigma_\gamma \in H^{\perp} \subseteq \hat{G}$,
  - $\eta(\gamma) = \gamma - \theta(\gamma)$.
- For any fixed $[s] \in G/H$, define
  - the following unimodular (hence L^∞) *weight function*:

$$\omega_{[s]}(\gamma) = \omega_{[s],\mathcal{D}}(\gamma) = \overline{\chi_s(\eta_{\mathcal{D}}(\gamma))},$$

  - the multiplier $M_{[s]}$ on $L^2(\hat{G})$, for any $F \in L^2(\hat{G})$,

$$M_{[s]}F(\gamma) = M_{[s],\mathcal{D}}F(\gamma) = F(\gamma)\omega_{[s],\mathcal{D}}(\gamma), \text{ and}$$

  - the shift operator $T_{[s]}f$

$$T_{[s]}f = T_{[s],\mathcal{D}}f = f * \check{\omega}_{[s],\mathcal{D}},$$

"Benedetto & Benedetto" p-Adic Shifts

- We'll write $\mathbb{Q}_p$ for both $\mathbb{Q}_p$ and $\hat{\mathbb{Q}}_p$, etc.
- Also $\mathcal{D}$ is fixed to be
  $I_p = \{\pi_N p^N + \cdots + \pi_{-1} p^{-1} : N \in \mathbb{Z}^-, \pi_i \in \{0, \ldots, p_1\}\}$.

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Definition

For any fixed $[\alpha] \in \mathbb{Q}_p/\mathbb{Z}_p$ (with $\alpha \in I_p$), define

- $\omega_{[\alpha]}(\gamma) = e^{2\pi i \{\alpha(\gamma - \{\gamma\}_p)\}_p}$.

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- $\omega_{[\alpha]}(\gamma) = e^{2\pi i \{\alpha(\gamma - \{\gamma\}_p)\}_p}$.
- $M_{[\alpha]} F(\gamma) = F(\gamma) \omega_{[\alpha]}(\gamma)$.
- $T_{[\alpha]} f = \left(M_{[\alpha]} \hat{f} \right)^{\vee}$.

Note in particular that the restriction of each $\omega_{[\alpha]}$ to $\mathbb{Z}_p$ is the restriction of the character defined by α to $\mathbb{Z}$.

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Theorem (K. 2013)

For $\beta \in I_p$, $N \in \mathbb{Z}$,

$$\begin{cases} \operatorname{supp} T_{[\beta]} \mathbb{1}_{p^N \mathbb{Z}_p} = \beta + \mathbb{Z}_p & ; \quad N \leq 0 \\ \operatorname{supp} T_{[\beta]} \mathbb{1}_{p^N \mathbb{Z}_p} = \beta + p^N \mathbb{Z}_p & ; \quad N > 0 \end{cases} .$$

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(Note that both sets of shifts give rise to the same Haar wavelets and generalize $\mathbb{R}$ shifts.)

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Shift-Invariant Spaces

Definition

A *shift-invariant space* $\mathcal{S}$ is a space of functions over G such that for any $h \in H \subset G$, $f \in \mathcal{S} \iff T_h f \in \mathcal{S}$. A *principal shift-invariant space* is a shift-invariant space that may be written for some $\psi \in \mathcal{S}$ as

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- For $G = \mathbb{R}^d$, H is classically $\mathbb{Z}^d$.
- Example, Paley-Wiener Space (band-limited functions, canonical ex. of translation invariant):

$$PW(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]\}$$

- Example, V_0 in a multi resolution analysis.

LCAG Bracket [Hernández, Sikic, Weiss, Wilson 2010]

$(G, +)$ LCAG, $T : g \to T_g$ unitary representation of G on Hilbert space $\mathcal{H}$

Definition

A unitary representation T is *dual integrable* if there exists a "bracket"

$$[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \to L^1(\widehat{G}, d\alpha)$$

s.t. $\forall \varphi, \psi \in \mathcal{H}, \forall g \in G$,

$$\langle \varphi, T_g \psi \rangle_{\mathcal{H}} = \int_{\widehat{G}} [\varphi, \psi](\alpha) \overline{\alpha}(g) d\alpha.$$

LCAG Shift-Invariant Systems

Theorem (Hernández, Sikic, Weiss, Wilson 2010)

If $\psi \in \mathcal{H}\backslash\{0\}$, *define* $\langle\psi\rangle = \overline{\{T_g\psi\}_{g\in G}}^{\mathcal{H}}$.

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- *If* $\psi \in \mathcal{H}\backslash\{0\}$, *then* $\{T_g\psi\}_{g\in G}$ *is an o.n.b. of* $\langle\psi\rangle \iff [\varphi,\psi](\alpha) = 1$ *a.e.* $\alpha \in \hat{G}$.

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- *If* $\psi \in \mathcal{H}\backslash\{0\}$, *then* $\{T_g\psi\}_{g\in G}$ *is a Riesz basis of* $\langle\psi\rangle$ *with constants* $A, B \iff A \leq [\varphi,\psi](\alpha) \leq B$ *a.e.* $\alpha \in \hat{G}$.

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- *If* $\psi \in \mathcal{H}\backslash\{0\}$, *then* $\{T_g\psi\}_{g\in G}$ *is a frame of* $\langle\psi\rangle$ *with constants* $A, B \iff A \leq [\varphi,\psi](\alpha) \leq B$ *a.e.* $\alpha \in \Omega_\psi$, *where* $\Omega_\psi = \{\alpha \in \widehat{G} : [\psi,\psi](\alpha) > 0\}$.

Prüfer Group Shift-Invariant Systems

Proposition (K. 2013)

Let $\Psi : \mathbb{Q}_p/\mathbb{Z}_p \to U(L^2(\mathbb{Q}_p))$ *be defined as* $\Psi([\alpha]) = T_{[a]}$. *This is a unitary representation. Further, this representation is dual integrable with bracket* $[\cdot,\cdot] : L^2(\mathbb{Q}_p) \times L^2(\mathbb{Q}_p) \to L^1(\mathbb{Z}_p)$ *mapping*

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- At this point not clear if the bracket can be generalized to "classical" shifts.

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Classical Wiener Inversion

Theorem (Wiener 1933)

If $F \in A(\mathbb{T})$ *vanishes nowhere then* $\frac{1}{F} \in A(\mathbb{T})$.

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Theorem (Sjöstrand 1994 & 1995)

If K_σ is a pseudodifferential operator with (Weyl or Kohn-Nirenberg) symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and if K_σ is invertible with respect to composition of operators on $L^2(\mathbb{R}^d)$, then $K_\sigma^{-1} = K_\tau$ for some $\tau \in M^{\infty,1}(\mathbb{R}^{2d})$.

Wiener-Type Inversion for ΨDO over LCAG

Theorem (Gröchenig and Strohmer 2007)

Let ν be an admissible weight. If $\sigma \in M^{\infty,1}_{\nu\circ\mathcal{J}^{-1}}(G \times \hat{G})$ and if K_σ is invertible on $L^2(G)$, then $(K_\sigma)^{-1} = K_\tau$ for some $\tau \in M^{\infty,1}_{\nu\circ\mathcal{J}^{-1}}(G \times \hat{G})$.

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The proof uses methods from time-frequency analysis rather than hard analysis techniques. They also generalize a result of [Baskakov 1990] about invertible matrices with ℓ^1-bounded diagonals.

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For $G \cong \mathbb{R}^d \times G_0$, we define the *Gaussian* φ, over $(x_1, x_2) \in \mathbb{R}^d \times G_0$ to be

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Example: Over $\mathbb{Q}_p$,

$$\mathbb{1}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p}^2 = \hat{\mathbb{1}}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p} * \mathbb{1}_{\mathbb{Z}_p}.$$

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- Tools from harmonic analysis can be used to approach many problems.
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- Thinking about T_α and $T_{[\alpha]}$ can sometimes help. (E.g., "square root" MRA expressed geometrically.)
- Zak-transform-like bracket for T_α?
- A fuller exploration of "continuous" systems (some preliminary work done) – including extended metaplectic representation?

Thanks!