Haar multiresolution analysis and Haar bases on the ring of rational adeles

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Outline

Real Haar wavelet basis

For each prime p the system of functions

$$\{p^{n/2}\psi^{(i)}(p^n x - b),\ x \in \mathbb{R} \,:\, n, b \in \mathbb{Z},\ i = 1, \ldots, p-1\}$$

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forms an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$. Here $\psi^{(i)} = \psi^{(i)}_{\infty,p}$ is the *i*th real Haar wavelet function:

$$\psi^{(i)}_{\infty,p}(x) = \begin{cases} \zeta^{ij}, & \text{if } \frac{j}{p} \le x < \frac{j+1}{p},\ j = 0, \dots, p-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$ is a primitive pth root of unity.

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The set of translations is a discrete group isomorphic to $\mathbb{Z}$.

Real Haar MRA (Meyer, Mallat, 1980's)

For each n let us define a closed subspace of $L^2(\mathbb{R})$ by

$$V_n = \text{span}\{\chi_M \ : \ M = [\frac{b}{p^n}, \frac{b+1}{p^n}), \ b \in \mathbb{Z}\}.$$

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Then the sequence $\{V_n\}_{n\in\mathbb{Z}}$ forms an MRA in $L^2(\mathbb{R})$ with $\varphi = \varphi_\infty = \chi_{[0,1)}$ as a scaling function:

- $V_n \subset V_{n+1}$ for all n,
- $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}), \quad \bigcap_{n\in\mathbb{Z}} V_n = \{0\}$,
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It follows that $L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n$ where $W_n = V_{n+1} \ominus V_n$.

The field $\mathbb{Q}_p$ of p-adic numbers

By definition $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r. to the topology in which x is close to y if $x - y$ is divided by a large power of p.

Each non-zero $x \in \mathbb{Q}_p$ is uniquely represented in the form

$$x = \sum_{i=k}^{\infty} x_i p^i \quad \text{where} \quad 0 \leq x_i \leq p - 1, \; x_k \neq 0.$$

The fractional part of x is defined by $\{x\}_p = \sum_{i=k}^{-1} x_i p^i$ and the ring of p-adic integers by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \{x\}_p = 0\}$.

The field $\mathbb{Q}_p$ is locally compact whereas the ring $\mathbb{Z}_p$ is compact. Denote by μ_p the normalized Haar measure on $\mathbb{Q}_p$.

p-Adic Haar wavelet basis (Kozyrev, 2002)

Similarly, in the p-adic case the system of functions

$$\{p^{n/2}\psi^{(i)}(p^{-n}x-b),\ x\in\mathbb{Q}_p\ :\ n\in\mathbb{Z},\ b\in I_p,\ i=1,\dots,p-1\}$$

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Note. The set of translations here is a discrete set, not a group. There are no non-trivial discrete subgroups of $\mathbb{Q}_p$.

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Definition

Let K be a field.

A function $|\cdot| : K \to \mathbb{R}$ is an absolute value or norm if

- $|x| \geq 0$ and $|x| = 0 \iff x = 0$,
- $|xy| = |x||y|$,
- $|x + y| \leq |x| + |y|$.

We have $|1| = 1$ and $|x| = |-x|$ for all $x \in K$.

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The norm is ultrametric if $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$. Otherwise it is archimedean.

In the ultrametric case every triangle in K is isosceles with respect to the metric $\rho(x, y) := |x - y|$.

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$$|x|_\infty = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{otherwise.} \end{cases}$$

- $K = \mathbb{Q}_p$: the p-adic norm $|\cdot|_p$ on $\mathbb{Q}_p$

$$|x|_p = p^{-k} \text{ where } x = p^k u, \quad u \in \mathbb{Z}_p^\times.$$

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  This norm is ultrametric.
- $K = \mathbb{Q}$: the restrictions of $|\cdot|_\infty$ and $|\cdot|_p$ to $\mathbb{Q}$. The fields $\mathbb{R}$ and $\mathbb{Q}_p$ are completions of $\mathbb{Q}$ with respect to them.

Ostrowski's Theorem (1916)

Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if

$$\{x \in K : |x|_1 < 1\} = \{x \in K : |x|_2 < 1\}.$$

In this case $|x|_1^c = |x|_2$ for all x where $c > 0$.

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Theorem Any non-trivial absolute value on the field $\mathbb{Q}$ is equivalent either to the real absolute value $|\cdot|_\infty$ or to the p-adic absolute value $|\cdot|_p$ where p is a prime.

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Classes of absolute values on $\mathbb{Q}$ are in 1-1 correspondence to the elements of $\mathbb{P} \cup \{\infty\}$ where $\mathbb{P}$ is the set of all primes.

We refer to $\mathbb{P} \cup \{\infty\}$ as the extended set of primes.

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Under the adele ring $\mathbb{A}$ we mean here the restricted direct product of the field $\mathbb{Q}_\infty = \mathbb{R}$ and all p-adic fields $\mathbb{Q}_p$ where $p \in \mathbb{P}$ with respect to their subrings $\mathbb{Z}_p$, i.e. the set of all

$$x = (x_\infty, \dots, x_p, \dots) \quad \text{where} \quad x_\infty \in \mathbb{R},\ x_p \in \mathbb{Q}_p,$$

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Denote by $\mathbb{A}^\times$ the multiplicative group of the ring $\mathbb{A}$. Then

$$x \in \mathbb{A}^\times \implies x_p \in \mathbb{Z}_p^\times \text{ for almost all } p \in \mathbb{P}.$$

Thus, $\mathbb{A}^\times$ is the restricted direct product of the group $\mathbb{R}^\times$ and all groups $\mathbb{Q}_p^\times$ with respect to their subgroups $\mathbb{Z}_p^\times$.

Adelic topology

From the definition of $\mathbb{A}$ it follows that

$$\mathbb{A} = \bigcup_S \mathbb{A}_S \quad \text{with} \quad \mathbb{A}_S = \mathbb{Q}_S \times \prod_{p \notin S} \mathbb{Z}_p$$

where $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$ and S runs over all finite subsets of the set $\mathbb{P} \cup \{\infty\}$ that contain ∞.

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Since the fields $\mathbb{Q}_\infty$ and $\mathbb{Q}_p$ are locally compact and the ring $\mathbb{Z}_p$ is compact for all $p \in \mathbb{P}$, the ring $\mathbb{A}$ is locally compact.

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Convergence: $x^{(n)} \to 0$ in $\mathbb{A}$ if and only if $x^{(n)} \in \mathbb{A}_S$ for some S and all n and $x_v^{(n)} \to 0$ for all $v \in \mathbb{P} \cup \{\infty\}$.

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Adelic topology is metrizable.

Field $\mathbb{Q}$ as a discrete subring of $\mathbb{A}$

Let $x \in \mathbb{Q}$. If $x = a/b$, then $x \in \mathbb{Z}_p$ for all $p \in \mathbb{P}$ not dividing b. Therefore $\mathbb{Q}$ can be embedded diagonally in $\mathbb{A}$

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Proof. It suffices to show that the set $F = [0,1) \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is a fundamental domain of $\mathbb{Q}$. This means that the translations of F by elements of $\mathbb{Q}$ are pairwise disjoint and form a partition of $\mathbb{A}$.

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Product formula

For each $x \in \mathbb{A}^\times$ set

$$\|x\|_{\mathbb{A}} = \prod_{v \in \mathbb{P} \cup \{\infty\}} |x_v|_v = |x_\infty|_\infty \prod_{p \in \mathbb{P}} |x_p|_p.$$

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The product is well defined because $|x_p|_p = 1$ for almost all p. Since all local norms are multiplicative, we have

$$\|xy\|_{\mathbb{A}} = \|x\|_{\mathbb{A}} \|y\|_{\mathbb{A}} \quad \text{for all} \quad x, y \in \mathbb{A}^\times.$$

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Proof. Let $x = \pm \prod_{p \in \mathbb{P}} p^{k_p}$ where $k_p = 0$ for almost all p. Then $|x|_\infty = \prod_{p \in \mathbb{P}} p^{k_p}$ and $|x|_p = p^{-k_p}$ for $p \in \mathbb{P}$. Thus, $\|x\|_{\mathbb{A}} = 1$. •

Haar measure on $\mathbb{A}$

Denote by $\mu_{\mathbb{A}}$ the Haar measure on $\mathbb{A}$ normalized by $\mu_{\mathbb{A}}(F) = 1$ where $F = [0,1) \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. Then $\mu_{\mathbb{A}}$ equals the product of the Lebesgue measure μ_∞ on $\mathbb{R}$ and all p-adic measures μ_p on $\mathbb{Q}_p$.

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Since $\mu_v(aX) = |a|_v \mu_v(X)$ for all $v \in \mathbb{P} \cup \{\infty\}$, $a \in \mathbb{Q}_v^\times$, $X \subset \mathbb{Q}_v$, it follows that given $X \subset \mathbb{A}$ we have

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$$\mu_{\mathbb{A}}(aX) = \|a\|_{\mathbb{A}} \mu_{\mathbb{A}}(X) \quad \text{for all} \quad a \in \mathbb{A}^\times.$$

In particular, the product formula implies that

$$\mu_{\mathbb{A}}(aX) = \mu_{\mathbb{A}}(X) \quad \text{for all} \quad a \in \mathbb{Q}^\times.$$

The measure $\mu_{\mathbb{A}}$ is regular. ‘

Dense subspace of $L^2(\mathbb{A})$

For a finite set $S \subset \mathbb{P} \cup \{\infty\}$ let

$$L_S = L^2(\mathbb{Q}_S) \otimes \mathbb{C}\varphi_{S'}$$

where $\varphi_{S'} = \bigotimes_{v \notin S} \varphi_v$ with $\varphi_\infty = \chi_{[0,1)}$ and $\varphi_p = \chi_{\mathbb{Z}_p}$ for $p \in \mathbb{P}$.

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If S contains ∞, then obviously L_S is a subspace of the space $L^2(\mathbb{A}_S) = L^2(\mathbb{Q}_S) \otimes L^2(\prod_{p \notin S} \mathbb{Z}_p)$.

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where $\varphi_{S'} = \bigotimes_{v \notin S} \varphi_v$ with $\varphi_\infty = \chi_{[0,1)}$ and $\varphi_p = \chi_{\mathbb{Z}_p}$ for $p \in \mathbb{P}$.
(For example, if $S = \emptyset$, then $L_S = \mathbb{C}\varphi$ where $\varphi = \bigotimes_v \varphi_v = \chi_F$.)

If S contains ∞, then obviously L_S is a subspace of the space $L^2(\mathbb{A}_S) = L^2(\mathbb{Q}_S) \otimes L^2(\prod_{p \notin S} \mathbb{Z}_p)$.

Theorem The union of L_S over all finite S is dense in $L^2(\mathbb{A})$.

Dense subspace of $L^2(\mathbb{A})$

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Theorem The union of L_S over all finite S is dense in $L^2(\mathbb{A})$.

Proof. Follows from the regularity of the measure $\mu_{\mathbb{A}}$. •

Tensor product of bases

For each $v \in \mathbb{P} \cup \{\infty\}$ let Ψ_v be an orthonormal basis in $L^2(\mathbb{Q}_v)$ such that $\varphi_v \in \Psi_v$. Set

$$\Psi = \{ \bigotimes_{v \in \mathbb{P} \cup \{\infty\}} \psi_v : \psi_v \in \Psi_v \text{ for all } v, \ \psi_v = \varphi_v \text{ for almost all } v \}.$$

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Proof. Obviously the set Ψ is orthonormal. Moreover, for any finite S it contains a basis of $L_S = \bigotimes_{v \in S} L^2(\mathbb{Q}_v) \otimes \mathbb{C}\varphi_{S'}$. So the closure of the linear span of Ψ contains the closure of the union of all spaces L_S, which equals $L^2(\mathbb{A})$ by above. •

Adelic dilations

Adelic dilations

For $r \in \mathbb{A}^\times$ set

$$D_r = T^{-1} H_r T$$

where H_r and T are defined by $H_r(x) = rx$, $x \in \mathbb{A}$, and by

$$T(x) = (x_\infty + \{x_{\text{fin}}\},\, x_{\text{fin}}), \quad x = (x_\infty,\, x_{\text{fin}}),$$

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We observe that $D_r(x)_{\text{fin}} = r_{\text{fin}} x_{\text{fin}}$ for all $x \in \mathbb{A}$, and if $r_{\text{fin}} = 1$, then D_r is reduced to the homothetic transformation of $\mathbb{Q}_\infty = \mathbb{R}$ with coefficient r_∞ with respect to $\{x_{\text{fin}}\}$ as an origin.

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$$D_1 = \text{id}_{\mathbb{A}}, \quad D_r^{-1} = D_{r^{-1}}, \quad D_r D_s = D_{rs}.$$

Transformations D_r will be treated as adelic dilations.

Partitions of $\mathbb{A}$

For each $r \in \mathbb{A}^\times$ let us define a partition of $\mathbb{A}$ by

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Below we set $\|r\| = (\dots, |r_v|_v, \dots)$ and for $a, b \in \mathbb{A}^\times$ say that a divides b if $(b/a)_\infty \in \mathbb{Z}$ and $(b/a)_p \in \mathbb{Z}_p$ for all $p \in \mathbb{P}$.

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Lemma The following statements hold:

- $P(s)$ is a refinement of $P(r)$ if and only if $\|r\|$ divides $\|s\|$,
- partition refining every partition $P(r)$ equals the partition of $\mathbb{A}$ into points,
- partition that is refined by every partition $P(r)$ equals the partition of $\mathbb{A}$ with the only element $\mathbb{A}$.

Proof of Lemma

For each $r \in \mathbb{A}^{\times}$ set

$$P_0(r) = \{T^{-1}(M) : M \in P(r)\}.$$

Then $P_0(r)$ equals the partition of $\mathbb{A}$ into the sets

$$M = M_\infty \times \prod_{p \in \mathbb{P}} M_p, \quad M_p = \mathbb{Z}_p \text{ for almost all } p$$

where $M_\infty \subset \mathbb{R}$ is a right for $r_\infty > 0$ and left for $r_\infty < 0$ semi-segment of length $|r_\infty|_\infty^{-1}$ with the ends in the set $r_\infty^{-1}\mathbb{Z}$, and $M_p \subset \mathbb{Q}_p$ is a coset by the group $r_p^{-1}\mathbb{Z}_p$ for $p \in \mathbb{P}$.

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Now Lemma follows from the corresponding statemente on $P_0(r)$'s which are obvious.

Family of closed spaces in $L^2(\mathbb{A})$

For $r \in \mathbb{A}^\times$ let us define a closed subspace of $L^2(\mathbb{A})$ by

$$V(r) = \text{span}\{\chi_M : \ M \in P(r)\}.$$

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Theorem 1 The following statements hold:

- $V(r) \subset V(s)$ if and only if $\|r\|$ divides $\|s\|$.
- $\overline{\bigcup_{r \in \mathbb{A}^\times} V(r)} = L^2(\mathbb{A}), \quad \bigcap_{r \in \mathbb{A}^\times} V(r) = \{0\},$
- $V(rs) = \{f^{D_s} : f \in V(r)\}$ for all r, s.
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Proof. Follows from Lemma and the definitions. •

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We treat the family $\{V_r\}_{r\in\mathbb{A}^\times}$ as the adelic pseudo Haar MRA.

Elementary adelic dilations

For $p \in \mathbb{P}$ let us define two invertible adeles by

$$r(\infty, p) = (p, \dots, 1, \dots), \quad r(\mathrm{fin}, p) = (1, \dots, p^{-1}, \dots).$$

Then obviously, $\|r(t,p)\|_{\mathbb{A}} = p$ where $t \in \{\infty, \mathrm{fin}\}$.

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We call this transformation of $\mathbb{A}$ an elementary adelic dilation.

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If $t = \infty$, then $D_{t,p}$ is reduced to the homothetic transformation of $\mathbb{Q}_\infty = \mathbb{R}$ with coefficient p w.r. to $\{x_{\mathrm{fin}}\}$ as an origin, while if $t = \mathrm{fin}$, then it is reduced to the division by p on $\mathbb{Q}_p$ and to the translation by $\{(p^{-1}x)_{\mathrm{fin}}\} - \{x_{\mathrm{fin}}\}$ on $\mathbb{Q}_\infty$ (changing the origin).

Strategies

Definition We say that a sequence $\mathfrak{r} = \{r_n\}_{n\in\mathbb{Z}}$ of invertible adeles is a strategy if for each $n \in \mathbb{Z}$ there exist $t_n \in \{\infty, \mathrm{fin}\}$ and $p_n \in \mathbb{P}$ such that

- $r_{n+1}/r_n = r(t_n, p_n)$ for all $n \in \mathbb{Z}$,
- the sets $\{n \in \mathbb{Z}_{>0} : \ t_n = \infty\}$ and $\{n \in \mathbb{Z}_{<0} : \ t_n = \infty\}$ are infinite,
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The sequences $\{t_n\}$ and $\{p_n\}$ are uniquely determined by $\mathfrak{r}$. Conversely, $\mathfrak{r}$ is uniquely determined by $\{t_n\}$, $\{p_n\}$ and r_0.

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It is easy to see that strategies do exist.

MRA associated with a strategy

Theorem 2 For a strategy $\{r_n\}_{n\in\mathbb{Z}}$ set $V_n = V(r_n)$. Then

- $V_n \subset V_{n+1}$ for all n,
- $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{A})$, $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$,
- $V_{n+1} = \{f^{D_{t_n,p_n}} : f \in V_n\}$ for all n,
- $\{\varphi^{D_{r_0},b} : b \in \mathbb{Q}\}$ is an orthogonal basis of V_0.

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Proof. The definition of strategy implies that $\|r_n\|$ divides $\|r_{n+1}\|$ for all n and given $r \in \mathbb{A}^\times$ there exist integers $n_1 < n_2$ such that $\|r\|$ is divided by $\|r_{n_1}\|$ and divides $\|r_{n_2}\|$. Apply Theorem 1. •

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If $r_0 = 1$, then $\{V_n\}_{n\in\mathbb{Z}}$ is an MRA in $L^2(\mathbb{A})$ w.r. to the sequence of dilations $\{D_{t_n,p_n}\}_{n\in\mathbb{Z}}$ and with φ as a scaling function.

Generating Haar wavelets

Denote by $\psi^{(i)}_{\infty,p}$ and $\psi^{(i)}_{\mathrm{fin},p}$ the ith real and ith p-adic Haar wavelet functions. Then given $t \in \{\infty, \mathrm{fin}\}$ and $p \in \mathbb{P}$,

$$\psi^{(i)}_{t,p} \in L^2(\mathbb{Q}_v) \quad \text{for all} \quad i = 1, ..., p-1$$

with $v = \infty$ if $t = \infty$ and $v = p$ if $t = \mathrm{fin}$.

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$$\Psi_{t,p} = \{\psi^{(i)}_{t,p} \otimes \varphi_{v'} : \; i = 1, \ldots, p-1\}$$

where $\varphi_{v'}$ is the tensor product of φ_w over all w other than v.

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Elements of the sets $\Psi_{t,p}$ are treated as generating adelic Haar wavelets. We have $|\Psi_{t,p}| = p - 1$ and $\mathrm{supp}(\psi) = F$ for $\psi \in \Psi_{t,p}$.

Wavelet basis associated with a strategy

Theorem 3 For any strategy $\mathfrak{r} = \{r_n\}_{n\in\mathbb{Z}}$ the set of functions

$$\Psi(\mathfrak{r}) = \bigcup_{n\in\mathbb{Z}} \{\|r_n\|_{\mathbb{A}}^{1/2} \psi^{D_{r_n}, b} : \psi \in \Psi_{t_n, p_n},\ b \in \mathbb{Q}\}$$

forms an orthonormal basis in the Hilbert space $L^2(\mathbb{A})$.

Wavelet basis associated with a strategy

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forms an orthonormal basis in the Hilbert space $L^2(\mathbb{A})$.

Note. By definition any function belonging to $\Psi(\mathfrak{r})$ equals

$$\|r_n\|_{\mathbb{A}}^{1/2} \psi(D_{r_n}(x) - b), \quad x \in \mathbb{A},$$

where ψ is a generating Haar wavelet. However if $r_0 = 1$, then D_{r_n} is the composition of D_{t_i,p_i} with $0 \le i \le n-1$ for $n > 0$, and the inverse to the composition of D_{t_i,p_i} with $n \le i \le -1$ for $n < 0$. Thus the above function can be treated as a wavelet one.

Sketch of proof

By Theorem 2 it suffices to verify that for each $n \in \mathbb{Z}$ the set

$$\Psi_n(\mathfrak{r}) = \{||r_n||_{\mathbb{A}}^{1/2} \psi^{D_{r_n}, b} : \psi \in \Psi_{t_n, p_n}, \; b \in \mathbb{Q}\}$$

is an orthonormal basis of the space $V_{n+1} \ominus V_n$.

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The latter means that $\{\psi_{t,p}^{(i)} : \ i = 0, \dots p - 1\}$ is an orthonormal basis of span$\{\chi_{M_j} : j = 0, \dots p - 1\}$ where $M_j = [j/p, (j + 1)/p)$ if $t = \infty$ and $M_j = j + p\mathbb{Z}_p$ if $t = \text{fin}$, which is obvious.

Papers I

S. Evdokimov,
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