General notations: We consider an algebraically closed field $\mathbb{E}$ of characteristic zero which is either $\mathbb{C}$ or a complete ultrametric field $\mathbb{K}$. We denote by $\mathcal{A}(\mathbb{E})$ the $\mathbb{E}$-algebra of entire functions in $\mathbb{E}$, by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in $\mathbb{E}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{E})$ and by $\mathbb{E}(x)$ the field of rational functions.

Given $a \in \mathbb{K}$ and $R > 0$, we denote by
$d(a, R)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$ and by
$d(a, R^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$.

First topic: Zeros of the derivative of a p-adic meromorphic function

Results to due to Kamal Boussaf, Jacqueline Ojeda, Jean-Paul Bézivin and A. Escassut

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$. According to classical notations, we set $|f|(r) = \sup\{|f(x)| \mid |x| \leq r\}$. We know that

$$|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n = \lim_{|x| \to r,\ |x| \neq r} |f(x)|.$$

That notation defines an absolute value on $\mathcal{A}(\mathbb{K})$ and has continuation to $\mathcal{M}(\mathbb{K})$ as
$|\frac{f}{g}|(r) = \frac{|f|(r)}{|g|(r)}$ with $f,\ g \in \mathcal{A}(\mathbb{K})$.

Given $f,\ g \in \mathcal{A}(\mathbb{K})$, we denote by $W(f, g)$ the Wronskian of f and g.

In the paper by Kamal Boussaf, Jacqueline Ojeda and me, the following Theorems 1 and 2 are proven:

Theorem 1: *Let f, g be entire functions on $\mathbb{K}$ such that $W(f, g)$ is a non-identically zero polynomial. Then both f, g are polynomials.*

Theorem 2 is an easy consequence of Theorem 1:

Theorem 2: *Let f be a transcendental meromorphic function on $\mathbb{K}$ having finitely many multiple poles. Then f' takes every value infinitely many times.*

Theorem 2 led us to the following conjecture:

Conjecture: *Let f be a meromorphic function on $\mathbb{K}$ such that f' has finitely many zeros. Then f is a rational function.*

Definition and notations: Let $f \in \mathcal{M}(\mathbb{K})$. For each $r > 0$, we denote by by $\theta_f(r)$ the number of zeros of f in $d(0, r)$, taking multiplicity into account and set $\tau_f(r) = \theta_{\frac{1}{f}}(r)$. Similarly, we denote by $\psi_f(r)$ the number of multiple zeros of f in $d(0, r)$, each counted with its multiplicity and we set $\phi_f(r) = \psi_{\frac{1}{f}}(r)$.

A function h from $[1, +\infty[$ to $\mathbb{N}$ will be said *to have fine upper bound* if for some $d \in \mathbb{N}$, h satisfies $h(r) \leq r^d$ in $[1, +\infty[$.

Theorem 3: *Let f be a meromorphic function on $\mathbb{K}$ such that, for some $d \in \mathbb{N}$, ϕ_f has fine upper bound. If f' has finitely many zeros, then f is a rational function.*

Corollary 1: *Let f be a meromorphic function on $\mathbb{K}$ such that, for some $d \in \mathbb{N}$, ϕ_f has fine upper bound. If for some $b \in \mathbb{K}$ $f' - b$ has finitely many zeros, then f is a rational function.*

Corollary 2: *Let f be a transcendental meromorphic function on $\mathbb{K}$ such that τ_f has fine upper bound. Then $f^{(k)}$ takes every value in $\mathbb{K}$ infinitely many times, for each $k \in \mathbb{N}^*$.*

Corollary 3: *Let h be a transcendental entire function on $\mathbb{K}$ and $P \in \mathbb{K}[x]$. The differential equation $y'h = yP$ admits no transcendental entire solution f, such that ψ_f has fine upper bound.*

According to the p-adic Hayman conjecture, for every $n \in \mathbb{N}^*$ $f'f^n$ takes every non-zero value infinitely many times. Here Theorem 3 has an immediate application to that conjecture in the cases $n = 1$ or $n = 2$ which are not yet solved, except with additional hypotheses.

Corollary 4: *Let f be a meromorphic function on $\mathbb{K}$. Suppose that τ_f has fine upper bound. If $f'f^n - b$ has has finitely many zeros for some $b \in \mathbb{K}$, with $n \in \mathbb{N}$ then f is a rational function.*

Remark: Using Corollary 7 to study zeros of $f' + bf^2$ that are not zeros of f is not so immediate, as done in Theorems 3, 4, 5 [3], because of residues of f at poles of order 1.

Theorem 4: *Let f be a transcendental meromorphic function on $\mathbb{K}$ such that θ_f has fine upper bound. Then for every $b \in \mathbb{K}$, $b \neq 0$, $f' - b$ has infinitely many zeros.*

Corollary 5: *Let f be a transcendental meromorphic function on $\mathbb{K}$ having no residue different from 0. If θ_f has fine upper bound, then f takes every value $b \in K$ infinitely many times.*

Among various lemmas, we use the following:

Lemma A: *Let $f \in \mathcal{M}(d(0, R^-))$. For each $n \in \mathbb{N}$, and for all $r \in]0, R[$, we have*

$$|f^{(n)}|(r) \leq |n!| \frac{|f|(r)}{r^n}.$$

Notation: For each $n \in \mathbb{N}^*$, we set
$\lambda_n = \max\{\frac{1}{|k|},\ 1 \leq k \leq n\}$.

Remark: For every $n \in \mathbb{N}^*$, we have $\lambda_n \leq n$ because $k|k| \geq 1\ \forall k \in \mathbb{N}$. The equality holds for all n of the form p^h.

Second topic: Complex and p-adic meromorphic functions $f'P'(f),\ g'P'(g)$ sharing a small function

Results to due to Kamal Boussaf, Jacqueline Ojeda and A. Escassut

Now let $a \in \mathbb{K}$ and $R > 0$. We denote by $\mathcal{A}(d(a, R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(a, R^-)$ i.e. the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a, R^-)$ and we denote by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$ - subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n\in\ \mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

A polynomial $P \in \mathbb{E}[X]$ is called *a polynomial of uniqueness for a family of functions $\mathcal{F} \subset \mathcal{M}(\mathbb{E})$ or $\mathcal{F} \subset \mathcal{M}(d(a, R^-))$* if given any two functions $f,\ g \in \mathcal{F}$, the equality $P(f) = P(g)$ implies $f = g$.

We say that two functions $f,\ g \in \mathcal{M}(\mathbb{E})$ or $\phi,\ \psi \in \mathcal{M}((a, R^-))$ *share a function α, counting multiplicities* if $f - \alpha$ and $g - \alpha$ have the same zeros, with the same order. Particularly, α may be a constant.

Let $f,\ g$ be two meromorphic functions that belong to $\mathcal{M}(\mathbb{E})$ or $\mathcal{M}_u(d(a,R^-))$ let $P \in \mathbb{E}[X]$ be such that $P'(X)$ is of the form $X^n \prod_{j=2}^{l}(X-a_j)^{k_j}$ and assume that $f'P'(f)$ and $g'P'(g)$ share a small meromorphic function α, counting multiplicities. Following many papers on this kind of problem both in complex analysis and in p-adic analysis, previously we gave general conditions on the polynomial P in order to assure that $f'P'(f) = g'P'(g)$. Next, we showed that if $n \geq \sum_{j=2}^{l} k_j + 3$ then we have $P(f) = P(g)$. Moreover, if $f,\ g$ belong to $\mathcal{M}(\mathbb{K})$, then we can conclude $P(f) = P(g)$ when $n \geq \sum_{j=2}^{l} k_j + 2$. Finally if P is a polynomial of uniqueness for the family of functions we consider, then we can conclude $f = g$. Here we want to propose a new condition on the polynomial P, derived from recent results in algebraic geometry, in order to prove that $P(f) = P(g)$, without assuming $n \geq \sum_{j=2}^{l} k_j + 3$ or $n \geq \sum_{j=2}^{l} k_j + 2$.

Our new conclusions derived from the following Theorems A and B below.

Definitions and notations: Let $\mathbb{F}$ be an algebraically closed field of characteristic 0, let $P,\ Q \in \mathbb{F}[x]$, let $a_i,\ 1 \leq i \leq l$ be the zeros of P' of respective order k_i and let $b_j,\ 1 \leq j \leq h$ be the zeros of Q' of respective order q_j, let $s = \deg(P)$ and $m = \deg(Q)$.
Let $F' = \{a_i \mid 1 \leq i \leq l,\ \ Q(b_j) \neq P(a_i) \forall j = 1, ..., h\}$ and let $F'' = \{b_j \mid 1 \leq j \leq s, P(a_i) \neq Q(b_j) \forall i = 1, ..., l\}$.

Theorem A was published in Ramajunan Journal, by Ta Thi Hoai An and me.

Theorem A: *Let $P,\ Q \in \mathbb{K}[x]$. If one of the following two statements holds,*

$$\sum_{a_i \in F'} k_i \geq s - m + 2 \text{ (resp. } \sum_{a_i \in \Delta} k_i \geq s - m + 3,\text{)}$$

$$\sum_{b_j \in F''} q_j \geq 2 \text{ (resp. } \sum_{b_i \in \Lambda} q_j \geq 3,\text{)}$$

and if two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}(d(a, R^-)))$) satisfy $P(f(x)) = Q(g(x)),\ x \in \mathbb{K}$, (resp. $x \in d(a, R^-)$) then both f and g are constant (resp. belong to $\mathcal{M}_b(d(a, R^-)))$).

On the field $\mathbb{C}$, we have results due to Ta Thi Hoai An and Nguyen Thi Ngoc Diep:

Proposition B: *Let $P,\ Q \in \mathbb{C}[X]$ satisfy one of the two following conditions:*

$$\sum_{a_i \in F'} k_i \geq s - m + 3.$$

$$\sum_{b_j \in F''} q_j \geq 3.$$

Then there is no non-constant function $f,\ g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) - Q(g(x)) = 0\ \forall x \in \mathbb{C}$.

Notation and definition: Henceforth, we assume that $a_1 = P(a_1) = 0$ and that $P(X)$ is of the form $X^n \prod_{i=2}^{l} (X - a_i)^{k_i}$ with $n \geq 2$. The polynomial P will be said *to satisfy Hypothesis (G)* if $P(a_i) + P(a_j) \neq 0\ \forall (i \neq j)$

Proposition 2 : *Let* $P \in \mathbb{K}[X]$ *satisfy Hypothesis (G) and* $n \geq 2$ *(resp.* $n \geq 3$*). If meromorphic functions* $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{M}(d(a, R^-))$*) satisfy* $P(f(x)) = P(g(x)) + C$ $(C \in \mathbb{K}^*)$, $\forall x \in \mathbb{K}$ *(resp.* $\forall x \in d(a, R^-)$*) then both* f *and* g *are constant (resp.* f *and* g *belong to* $\mathcal{M}_b(d(a, R^-))$*).*

Proposition 3: *Let* $P \in \mathbb{C}[X]$ *satisfy Hypothesis (G) and* $n \geq 3$*. If meromorphic functions* $f,\ g \in \mathcal{M}(\mathbb{C})$ *satisfy* $P(f(x)) = P(g(x)) + C$ $(C \in \mathbb{C}^*)$, $\forall x \in \mathbb{C}$ *then both* f *and* g *are constant.*

Nevanlinna functions and polynomials of uniqueness

In order to define small functions, we must briefly recall the definitions of the classical Nevanlinna theory in $\mathbb{C}$ and in $\mathbb{K}$. Here, for convenience, we will use notation long ago used in p-adic analysis in order to denote counting functions.

Let log be a real logarithm function of base > 1. Given $u \in \mathbb{R}_+^*$, we denote by $\log^+$ the real function defined as $\log^+(u) = \max(\log(u), 0)$.

Let $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Suppose first that f has no zero and no pole at 0. Let $r \in]0, +\infty[$ and let $\gamma \in \mathbb{E}$ (resp. let $\gamma \in d(0, R^))$. If f has a zero of order n at γ, we set $\omega_\gamma(h) = n$. If f has a pole of order n at γ, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we put $\omega_\gamma(f) = 0$

We denote by $Z(r, f)$ the *counting function of zeros of* f in $\mathbb{E}$ (resp. in $d(0, R^-)$), counting multiplicities, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|).$$

Similarly, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f in $\mathbb{E}$ (resp. in $d(0, R^-)$), ignoring multiplicities, and set

$$\overline{Z}(r, f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} (\log r - \log|\gamma|).$$

In the same way, we set $N(r, f) = Z\Big(r, \dfrac{1}{f}\Big)$ (resp. $\overline{N}(r, f) = \overline{Z}\Big(r, \dfrac{1}{f}\Big)$) to denote the *counting function of poles of* f in $\mathbb{E}$

or in $d(0, R^-)$), counting multiplicity (resp. ignoring multiplicity).

If f admits a zero of order s at 0, we can make a change of origin or count the zero at 0 by adding $s \log r$ and similarly, if f admits a pole at 0 of order s, we can make a change of origin or count the pole at 0 by adding $-s \log r$.

Let $f \in \mathcal{M}(\mathbb{C})$. Given $r > 0$, we set

$$m(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

and the function $T(r, f) = m(r, f) + N(r, f)$ is called *the characteristic function of f*.

Now, let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). We set $T(r, f) = \max(Z(r, f), N(r, f))$ and $T(r, f)$ is called *the characteristic function* of f again.

Let $f \in \mathcal{M}(\mathbb{E})$. A function $\alpha \in \mathcal{M}(\mathbb{E})$ is called *a small function with respect to f*, if it satisfies

$$\lim_{r \to +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0.$$

We denote by $\mathcal{M}_f(\mathbb{E})$ the set of small meromorphic functions with respect to f in $\mathbb{E}$ (it is easily checked that $\mathcal{M}_f(\mathbb{E})$ is subfield of $\mathcal{M}(\mathbb{E})$).

Similarly, let $f \in \mathcal{M}(d(a, R^-))$. A function $\alpha \in \mathcal{M}(d(a, R^-))$ is called *a small function with respect to f*, if it satisfies

$$\lim_{r \to R^-} \frac{T(r, \alpha)}{T(r, f)} = 0.$$

We denote by $\mathcal{M}_f(d(a, R^-))$ the set of small meromorphic functions with respect to f in $d(a, R^-)$ (similarly, $\mathcal{M}_f(d(a, R^-))$ is subfield of $\mathcal{M}(d(a, R^-)))$.

Remark: For simplicity, we have kept the same notation on $\mathbb{C}$ and on $\mathbb{K}$ for counting functions of zeros and poles of a meromorphic function.

Now, we must examine polynomials of uniqueness in order to give some sufficient conditions to get polynomials P such that, if $f'P'(f)$ and $g'P'(g)$ share a small meromorphic function, then $f = g$.

Notation: Let $P \in \mathbb{F}[x] \setminus \mathbb{F}$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c. We denote by $\Phi(P)$ its cardinal.

Theorem H was first proved by Julie Wang:

Theorem H: *Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2. Then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{c_1, c_2\} \geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Remark: If $\deg(P) = t$ then $\Phi(P) \leq t - 1$. If $\Phi(P) < l$, then $l \geq \Phi(P) + 2$.

We have the following results:

Theorem J: *Let* $P \in \mathbb{K}[x]$.

If $\Phi(P) \geq 2$ *then* P *is a polynomial of uniqueness for* $\mathcal{A}(\mathbb{K})$.

If $\Phi(P) \geq 3$ *then* P *is a polynomial of uniqueness for both* $\mathcal{A}_u(d(a, R^-))$ *and* $\mathcal{M}(\mathbb{K})$.

If $\Phi(P) \geq 4$ *then* P *is a polynomial of uniqueness for* $\mathcal{M}_u(d(a, R^-))$.

Theorem L: *Let* $P \in \mathbb{K}[x]$ *be of degree* $n \geq 6$ *and such that* P' *only has two distinct zeros, one of them being of order* 2*. Then* P *is a polynomial of uniqueness for* $\mathcal{M}_u(d(0, R^-))$.

Concerning the field $\mathbb{C}$, from various results due to Ta Thi Hoai An Julie Wang, Pitman Wong, Frank and Reinder and me, we have the following theorems:

Theorem M: *Let* $P \in \mathbb{C}[X]$ *be such that* P' *has exactly two distinct zeros* γ_1 *of order* c_1 *and* γ_2 *of order* c_2 *with* $\min\{c_1, c_2\} \geq 2$ *and* $\max(c_1, c_2) \geq 3$*. Then* P *is a polynomial of uniqueness for* $\mathcal{M}(\mathbb{C})$.

Theorem S: *Let* $P \in \mathbb{C}[X]$*. If* $\Phi(P) \geq 4$ *then* P *is a polynomial of uniqueness for* $\mathcal{M}(\mathbb{C})$.

The following Theorem T holds both on the field $\mathbb{K}$ and on $\mathbb{C}$ and is useful in the proofs of Theorems 1-10.

Theorem T: *Let* $Q(X) = (X - a_1)^n \prod_{i=2}^{l}(X - a_i)^{k_i} \in \mathbb{E}[x]$ *(*$a_i \neq a_j, \ \forall i \neq j$*) with* $l \geq 2$ *and* $n \geq \max\{k_2, .., k_l\}$ *and let* $k = \sum_{i=2}^{l} k_i$*. Let* $f, \ g \in \mathcal{M}(\mathbb{E})$ *be transcendental (resp.* $f, \ g \in \mathcal{M}_u(d(0, R^-))$*) such that the function* $\theta = f'Q(f)g'Q(g)$ *is a small function with respect to* f *and* g*. We have the following:*
If $l = 2$ *then* n *belongs to* $\{k, \ k+1, \ 2k, \ 2k+1, \ 3k+1\}$*.*
If $l = 3$ *then* n *belongs to* $\{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, .., 3k_l - k\}$*.*
If $l \geq 4$ *then* $n = k + 1$*.*
If θ *is a constant in* $\mathbb{K}$ *and if* $f, \ g \in \mathcal{M}(\mathbb{K})$ *then* $n = k + 1$*.*

Remark: We don't know any pair of meromorphic functions $f, \ g$ and a polynomial Q such that $f'Q(f)g'Q(g)$ is a small function with respect to f and g.

Sharing values problems for meromorphic functions

The problem of value sharing a small function by functions of the form $f'P'(f)$ was examined first when P was just of the form x^n. More recently, it was examined when P was a polynomial such that P' had exactly two distinct zeros, both in complex analysis and in p-adic analysis. In p-adic analysis we have the opportunity to use the Nevanlinna theory not only in the whole field $\mathbb{K}$ but also inside a disk $d(a, R^-)$. Actually solving a values sharing problem involving $f'P'(f), \ g'P'(g)$ requires to know polynomials of uniqueness P for meromorphic functions.

We first considered functions $f, \ g \in \mathcal{M}(\mathbb{K})$ or $f, \ g \in \mathcal{M}(d(a, R^-))$ and ordinary polynomials P: we only had to assume certain hypotheses on the multiplicity order of the zeros of P': that was published in Buletin des Sciences Mathematiques. Next we dealt with the same problem with functions in $\mathbb{C}$ (that just appeared in Indagationes). In those papers, we had to assume that $n \geq \sum_{j=2}^{l} k_j + 3$ (or $n \geq \sum_{j=2}^{l} k_j + 2$ when $f,\ g, \alpha \in \mathcal{M}(\mathbb{K})$). Here thanks to Propositions 2 and 3, we can replace that hypothesis by Hypothesis (G).

We can now state our main theorems.

Theorem 1: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{E})$ (resp. for $\mathcal{M}_u(d(a, R^-))$) satisfying Hypothesis (G). Let*

$$P' = bX^n \prod_{i=2}^{l} (X - a_i)^{k_i}$$

with $b \in \mathbb{E}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:

$$n \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

if $l = 2$, then $n \neq k,\ k + 1,\ 2k,\ 2k + 1,\ 3k + 1$,

if $l = 3$, then $n \neq \dfrac{k}{2},\ k + 1,\ 2k + 1,\ 3k_i - k\ \forall i = 2, 3$,

if $l \geq 4$, then $n \neq k + 1$.

Let $f,\ g \in \mathcal{M}(\mathbb{E})$ (resp. $f,\ g \in \mathcal{M}_u(d(a, R^-))$) be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem J, we have Corollary 1.1:

Corollary 1.1 *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and Hypothesis (G), let $P' = bX^n \prod_{i=2}^{l}(X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$$n \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

if $l = 3$, then $n \neq \frac{k}{2},\ k+1,\ 2k+1,\ 3k_i - k\ \forall i = 2, 3,$

if $l \geq 4$, then $n \neq k+1$. Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{20}}{20} - \frac{X^{19}}{19} - \frac{4X^{18}}{18} + \frac{4X^{17}}{17}$$

$$+\frac{6X^{16}}{16} - \frac{6X^{15}}{15} - \frac{4X^{14}}{14} + \frac{4X^{13}}{13} + \frac{X^{12}}{12} - \frac{X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X-1)^5(X+1)^4$ and

$$P(0) = 0,\ P(1) = \sum_{j=0}^{4} C_4^j (-1)^j \Big(\frac{1}{10+2j} - \frac{1}{9+2j}\Big),$$

$$P(-1) = -\sum_{j=0}^{4} C_4^j \Big(\frac{1}{10+2j} + \frac{1}{9+2j}\Big)$$

Consequently, we have $\Phi(P)=3$ and we check that Hypothesis (G) is satisfied. Now, let $f,\ g\in\mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Remark: In that example, we have $n=10,\ k=9$. Applying our previous work, a conclusion would have required $n\geq k+2=11$.

Corollary 1.2 *Let $P\in\mathbb{K}[X]$ satisfy $\Phi(P)\geq 4$ and Hypothesis (G), let $P'=bX^n\prod_{i=2}^{l}(X-a_i)^{k_i}$ with $b\in\mathbb{K}^*,\ k_i\geq k_{i+1},\ 2\leq i\leq l-1$ and let $k=\sum_{i=2}^{l}k_i$. Suppose P satisfies the following conditions:*

$$n\geq 10+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i),$$

$$n\neq k+1.$$

Let $f,\ g\in\mathcal{M}_u(d(a,R^-))$ and let $\alpha\in\mathcal{M}_f(d(a,R^-))\cap\mathcal{M}_g(d(a,R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Corollary 1.3 *Let* $P \in \mathbb{C}[X]$ *satisfy* $\Phi(P) \geq 4$ *and Hypothesis (G), let* $P' = bX^n \prod_{i=2}^{l}(X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$*,* $k_i \geq k_{i+1}$*,* $2 \leq i \leq l-1$ *and let* $k = \sum_{i=2}^{l} k_i$*. Suppose* P *satisfies the following conditions:*

$$n \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

$$n \neq k + 1.$$

Let $f,\ g \in \mathcal{M}(\mathbb{C})$ *and let* $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ *be non-identically zero. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *C.M., then* $f = g$*.*

Example: Let

$$P(X) = \frac{X^{24}}{24} - \frac{10X^{23}}{23} + \frac{36X^{22}}{22} - \frac{40X^{21}}{21} - \frac{74X^{20}}{20} + \frac{226X^{19}}{19}$$
$$-\frac{84X^{18}}{18} - \frac{312X^{17}}{17} + \frac{321X^{16}}{16} + \frac{88X^{15}}{15}$$
$$-\frac{280X^{14}}{14} + \frac{48X^{13}}{13} + \frac{80X^{12}}{12} - \frac{32X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X-2)^5(X+1)^4(X-1)^4$. Next, we have $P(2) < -134378$, $P(1) \in]-2,11; -2,10[$, $P(-1) \in]2,18; 2,19[$. Therefore, $P(0)$, $P(1)$, $P(-1)$, $P(2)$ are all distinct, hence $\Phi(P) = 4$. Moreover, Hypothesis (G) is satisfied.

Now, let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. let $f,\ g \in \mathcal{M}_u(d(a, R^-))$, resp. let $f,\ g \in \mathcal{M}(\mathbb{C})$) and let $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $\alpha \in \mathcal{M}(d(a, R^-))$, resp. let $\alpha \in \mathcal{M}(\mathbb{C})$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In that example, we have $n = 10,\ k = 13$. Applying Theorem 4 in the paper in Bulletin des Sciences or Theorem 1 in the paper in Indagationes, a conclusion would have required $n \geq k + 3 = 16$.

When $l = 2$, Hypothesis (G) is automatically satisfied. So, by Theorem H we also have Corollary 1.4.

And by Theorem M we also have Corollary 1.4

Corollary 1.4 *Let $P \in \mathbb{C}[X]$ be such that P' is of the form $bX^n(X - a_2)^k$ with $\min(k, n) \geq 2$ and $\max(n, k) \geq 3$. Suppose that P satisfies the further conditions:*

$n \geq 10 + \max(0, 5 - k)$,

$n \neq k + 1, 2k,\ 2k + 1,\ 3k + 1.$

Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: Thanks to Corollary 1.4, we can take $k = 8,\ n = 10$ which we couldn't do in the paper in Indagationes.

Example: Let $P(X) = \dfrac{X^6}{6} - \dfrac{2X^5}{5} + \dfrac{X^4}{4}$. Then $P'(X) = X^3(X - 1)^2$.

Given $f,\ g \in \mathcal{M}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

Theorem 2: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ satisfying Hypothesis (G), let*

$P' = bX^n \prod_{i=2}^{l}(X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*,\ l \geq 2,\ k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose P satisfies the following conditions:*

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

if $l = 2,$ *then* $n \neq k,\ k+1,\ 2k,\ 2k+1,\ 3k+1,$

if $l = 3,$ *then* $n \neq \frac{k}{2},\ k+1,\ 2k+1,\ 3k_i - k\ \forall i = 2,\ 3,$

if $l \geq 4,$ *then* $n \neq k+1.$

Let $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* α *be a Moebius function. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *C.M., then* $f = g$.

Corollary 2.1 *Let* $P \in \mathbb{K}[X]$ *satisfy* $\Phi(P) \geq 3$ *and Hypothesis (G), let* $P' = bX^n \prod_{i=2}^{l}(X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*,\ k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose P satisfies the following conditions:*

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

if $l = 3,$ *then* $n \neq \frac{k}{2},\ k+1,\ 2k+1,\ 3k_i - k\ \forall i = 2, 3.$

if $l \geq 4,$ *then* $n \neq k+1.$

Let $f,\ g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* α *be a Moebius function. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *C.M., then* $f = g$.

When α is a constant and $\mathbb{E} = \mathbb{K}$, we can simplify the conditions on n and k,

Theorem 3: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ satisfying Hypothesis (G), let*

$P' = bX^n \prod_{i=2}^{l} (X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose P satisfies the following conditions:*

$$n \geq 9 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \neq k + 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In Theorem 3 of the paper in Indagationes, we obtained $n \geq k + 2$. Here we just have $n \neq k + 1$ instead.

Corollary 3.1 *Let $P \in \mathbb{K}[X]$ satisfy $\Phi(P) \geq 3$ and Hypothesis (G), let* $P' = bX^n \prod_{i=2}^{l} (X - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $k_i \geq k_{i+1}$, $1 \leq i \leq l-1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose P satisfies the following conditions:*

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),$$

$$n \neq k + 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A, we have Corollary 3.2

Corollary 3.2 *Let $P \in \mathbb{K}[X]$ be such that P' is of the form $bX^n(X-a_2)^k$ with $k \geq 2$ and with $b \in \mathbb{K}^*$ and $k \leq n$. Suppose P satisfies the following conditions:*

$n \geq 9 + \max(0, 5-k)$,

$n \neq k+1$,

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In Corollary 3.2 of the paper published in Bulletin des Sciences, we obtained $n \geq k+2$ (with an additional condition that said $n \neq 2k+1,\ 3k+1$ but actually this is useless and just comes from a misprint). So, here we have an improvement with the hypothesis $n \neq k+1$ instead of $n \geq k+2$. Actually, since $k \leq n$, we obtain the additional hypothesis $n = k$.

Example: Let

$$P(X) = \frac{X^{19}}{19} - \frac{X^{18}}{18} - \frac{4X^{17}}{17} + \frac{4X^{16}}{16} + \frac{6X^{15}}{15} - \frac{6X^{14}}{14} - \frac{4X^{13}}{13}$$
$$+\frac{4X^{12}}{12} + \frac{X^{11}}{11} - \frac{X^{10}}{10}$$

We can check that $P'(X) = X^9(X-1)^5(X+1)^4$ and

$$P(0) = 0,\ P(1) = \sum_{j=0}^{4} C_4^j (-1)^j \Big(\frac{1}{9+2j} - \frac{1}{8+2j}\Big)$$

$$P(-1) = -\sum_{j=0}^{4} C_4^j \Big(\frac{1}{9+2j} + \frac{1}{8+2j}\Big)$$

Consequently, we can check that $\Phi(P) = 3$ and that Hypothesis (G) is satisfied. Now, let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathbb{K}^*$. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

For memory, here we can recall and summarize the following Theorems 4, 5 , 6, 7 and Corollaries 4.1 and 5.1. Theorem 4 is partially given in [4] and partilally given in [5]. Theorem 5 is given in [4]. Theorems 6 is given in [4] and Theorem 7 is given in [5]. We can not improve them since the inequality $n \geq k + 3$ is satisfied in each statement.

Theorem 4: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{E})$ (resp. for $\mathcal{M}_u(d(a, R^-))$) satisfying Hypothesis (G). Let P' be of the form $bX^n \prod_{i=2}^{l}(X - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$, satisfying: $n \geq l + 10$.*
Let $f,\ g \in \mathcal{M}(\mathbb{E})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(a, R^-))$) and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 4.1 *Let $P \in \mathbb{K}[X]$ satisfy $\Phi(P) \geq 3$ (resp. $\Phi(P) \geq 4$ and be such that P' is of the form $bX^n \prod_{i=2}^{l}(X - a_i)$ with $l \geq 3$ (resp. $l \geq 4$), $b \in \mathbb{K}^*$, satisfying $n \geq l + 10$.*
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f,\ g \in \mathcal{M}_u(d(a, R^-))$) and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 5: *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form*

$$P' = bX^n \prod_{i=2}^{l}(X - a_i) \text{ with } l \geq 3,\ b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function or a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 5.1 *Let $P \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 3$ and be of the form*

$$P' = bX^n \prod_{i=2}^{l}(X - a_i) \text{ with } l \geq 3,\ b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function or a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In Theorems 4 and 5 and Corollaries 4.1 and 5.1, it is useless to specify $n \neq k + 1$ and if $l = 3$, $n \neq \frac{k}{2},\ k + 1,\ 2k + 1,\ 3k_i - k...$ because these condition are automaticaly satisfied due to the hypotheses $ln \geq l + 9$ and $k_i = 1 \forall i$.

Theorem 6: *Let $f,\ g \in \mathcal{M}(\mathbb{E})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{E}) \cap \mathcal{M}_g(\mathbb{E})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function α C.M. and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{E})$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1}-1}{h^{n+2}-1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1}-1}{h^{n+2}-1}\Big)$. Moreover, if $\mathbb{E} = \mathbb{K}$ and if α is a constant or a Moebius function, then the conclusion holds whenever $n \geq 11$.*

Inside an open disk, we have a version similar to the general case in the whole field.

Theorem 7: *Let* $f,\ g \in \mathcal{M}_u(d(0, R^-))$*, and let* $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ *be non-identically zero. Let* $a \in \mathbb{K} \setminus \{0\}$*. If* $f' f^n(f - a)$ *and* $g' g^n(g - a)$ *share the function* α *C.M. and* $n \geq 12$*, then either* $f = g$ *or there exists* $h \in \mathcal{M}(d(0, R^-))$ *such that* $f = \dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1} - 1}{h^{n+2} - 1}\Big)h$ *and* $g = \dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1} - 1}{h^{n+2} - 1}\Big)$.

Remark: As noticed in [4], in Theorems 7 and 8, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K})$ (resp. let $h \in \mathcal{M}_u(d(0, R^-))$). Now, let us precisely define f and g as: $g = (\dfrac{n+2}{n+1})\Big(\dfrac{h^{n+1)} - 1}{h^{n+2} - 1}\Big)$ and $f = hg$. Then we can see that the polynomial $P(y) = \dfrac{1}{n+2}y^{n+2} - \dfrac{1}{n+1}y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.

Sharing values problems for analytic functions

First we can improve results given in Bulletin des Sciences concerning p-adic analytic functions. In the paper given for Proceedings of the 12th Conference on p-adic Functional Analysis we gave the following theorem 8:

Theorem 8: *Let $P(X) \in \mathbb{K}[X]$ be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(a,R^-))$), let $P'(X) = \prod_{i=1}^{l}(X-a_i)^{k_i}$ and let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. let $f,\ g \in \mathcal{A}_u(d(a,R^-))$) such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(,R^-)) \cap \mathcal{A}_g(d(a,R^-))$). If $\sum_{i=1}^{l} k_i \geq 2l+2$ then $f = g$. Moreover, if $f,\ g$ belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $\sum_{i=1}^{l} k_i \geq 2l+1$ then $f = g$.*

Corollary 8.1: *Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$, let $P'(X) = \prod_{i=1}^{l}(X-a_i)^{k_i}$ and let $f,\ g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$. If $\sum_{i=1}^{l} k_i \geq 2l+2$ then $f = g$. Moreover, if α is a constant and if $\sum_{i=1}^{l} k_i \geq 2l+1$ then $f = g$.*

Example: Let $\mathcal{E}$ be the algebraic equation:

$$X^{14}\Big(\frac{1}{14}-\frac{1}{13}\Big)-X^{12}\Big(\frac{1}{12}-\frac{1}{11}\Big)-\Big(\frac{1}{14}-\frac{1}{13}\Big)+\frac{1}{12}-\frac{1}{11}=0$$

and let $c \in \mathbb{K}$ be a solution of $\mathcal{E}$. Let

$$P(X)=\frac{X^{14}}{14}-\frac{cX^{13}}{13}-\frac{X^{12}}{12}+\frac{cX^{11}}{11}$$

Then we can check that $P'(X)=X^{10}(X-1)(X+1)(X-c)$, $P(1)=P(c)\neq 0$ and that $P(1)\neq 0$, $P(-1)\neq 0$, $P(1)+P(-1)=\frac{1}{7}-\frac{1}{6}$, and $P(-1)-P(1)=2c\Big(\frac{1}{11}-\frac{1}{13}\Big)$, hence $P(-1)\neq P(c)$. Consequently, $\Phi(P)=2$. Consequently, we can apply Corollary 8.1 and show that if $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K})\cap\mathcal{A}_g(\mathbb{K})$, then $f=g$.

Remark: Recall Hypothesis (F) due to H. Fujimoto. A polynomial Q is said to satisfy Hypothesis (F) if the restriction of Q to the set of zeros of Q' is injective. In the last example, we may notice that Hypothesis (F) is not satisfied by P.

Corollary 8.2: *Let $P(X)\in\mathbb{K}[X]$ be such that $\Phi(P)\geq 3$, let $P'(X)=\prod_{i=1}^{l}(X-a_i)^{k_i}$ and let $f,\ g\in\mathcal{A}_u(d(a,R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha\in\mathcal{A}_f(d(a,R^-))\cap\mathcal{A}_g(d(a,R^-))$. If $\sum_{i=1}^{l}k_i\geq 2l+2$ then $f=g$.*

Corollary 8.3: *Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$, (resp. $\Phi(P) \geq 3$) be such that $P'(X) = X^n \prod_{i=2}^{l}(X - a_i)$ with $l \geq 3$ and let $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}_u(d(a, R^-))$) be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $n \geq l + 3$ then $f = g$. Moreover, if $f,\ g$ belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $n \geq l + 2$ then $f = g$.*

Concerning complex analytic functions in $\mathbb{C}$, we can improve previous results.

Theorem 9: *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ satisfying Hypothesis (G), let*
$P' = bX^n \prod_{i=2}^{l}(X - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies $n \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i)$

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Proposition 3, we have Corollary 9.1:

Corollary 9.1 *Let $P \in \mathbb{C}[X]$ satisfy satisfying Hypothesis (G), let $P' = bX^n \prod_{i=2}^{l} (X - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $k_i \geq k_{i+1}$, $1 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies*

$$n \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^{l} \max(0, 4 - k_i),,$$

Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Proposition 2 we also have Corollary 9.2

Corollary 9.2 *Let $P \in \mathbb{C}[X]$ be such that P' is of the form $bX^n(X - a)^k$ with $\min(k, n) \geq 2$ and $\max(n, k) \geq 3$. Suppose that P satisfies $n \geq 5 + \max(0, 5 - k)$,*

Let $f,\ g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{11}}{11} + \frac{5X^{10}}{10} + \frac{10X^9}{9} + \frac{10X^8}{8} + \frac{5X^7}{7} + \frac{X^6}{6}.$$

Then $P'(X) = X^5(X+1)^5$. We can apply Corollary 9.2: given $f,\ g \in \mathcal{A}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

Remark: If we had applied Theorem 1 in the paper in Indagationes, with $k = 5$, we should have taken $n \geq k + 2$, hence $n \geq 7$.

When all k_i are equal to 1, we can obtain a better formulation:

Theorem 10: *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ satisfying Hypothesis (G), such that P' is of the form*

$bX^n \prod_{i=2}^{l}(X - a_i)$ *with* $l \geq 3$, $b \in \mathbb{C}^*$, *satisfying* $n \geq l + 5$.

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Proposition 3, we have Corollary 10.1:

Corollary 10.1 *Let $P \in \mathbb{C}[X]$ satisfy $\Phi(P) \geq 4$ and satisfy Hypothesis (G) and be such that P' is of the form $bX^n \prod_{i=2}^{l}(X - a_i)$ and $b \in \mathbb{C}^*$, satisfying $n \geq l + 5$.*

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = \dfrac{X^{13}}{13} - \dfrac{2X^{12}}{12} - \dfrac{X^{11}}{11} + \dfrac{2X^{10}}{10}$. Then $P'(X) = X^9(X-1)(X+1)(X-2)$. We check that:

$P(0) = 0, \quad P(1) = \dfrac{1}{13} - \dfrac{2}{12} - \dfrac{1}{11} + \dfrac{2}{10}$,

$P(-1) = \dfrac{1}{13} + \dfrac{2}{12} - \dfrac{1}{11} - \dfrac{2}{10} \neq 0,\ P(1)$. Further, we notice that and $|P(1)| < 1,\ |P(-1)| < 1$.

Finally, $P(2) = \dfrac{2^{13}}{13} - \dfrac{2^{13}}{12} - \dfrac{2^{11}}{11} + \dfrac{2^{11}}{10} = -\dfrac{72704}{2145} > 33$ hence $P(2) \neq 0, P(1), P(-1)$. Then $\Phi(P) = 4$.

So, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ and it clearly satisfies Hypothesis (G). Moreover, we have $n = 9$, $l = 4$, so we can apply Corollary 10.1. Given $f,\ g \in \mathcal{A}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}(\mathbb{C})$ C.M., we have $f = g$.